上海交通大學
SHANGHAI JIAO TONG UNIVERSITY
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Fourier Series

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LTI Systems

$$x_k[n] \rightarrow y_k[n] \implies \sum_k a_k x_k[n] \rightarrow \sum_k a_k\, y_k[n]$$

- **If we can find a set of "basic" signals, such that**
  - a rich class of signals can be **represented as linear combinations** of these basic (building block) signals.
  - the **response** of LTI Systems to these basic signals are both **simple and insightful**

- **Candidate sets of "basic" signals**
  - Unit impulse function and its delays: $\delta(t)/\delta[n]$
  - Complex exponential/sinusoid signals: e^{st}/z^n

Candidate Sets of Basic Signals

- **Time domain:** $\delta(t)/\delta[n]$

$x(t)=\int_{-\infty}^{+\infty}x(\tau)\delta(t-\tau)\,d\tau$	$x(n)=\sum_{k=-\infty}^{+\infty}x[k]\delta[n-k]$
$\delta(t)\to h(t)$ $x(t)\to y(t)=x(t)*h(t)$	$\delta[n]\to h[n]$ $x[n]\to y[n]=x[n]*h[n]$

- **Frequency domain:** e^{jwt}/e^{jwn} and e^{st}/z^{n}

$x(t)=\int_{\omega}?\,e^{j\omega t}\,d\omega$	$x[n]=\int_{\omega}?\,e^{j\omega n}\,d\omega$
$e^{j\omega t}\to H(j\omega)e^{j\omega t}$ $x(t)\to y(t)=\int_{\omega}?\,H(j\omega)e^{j\omega t}\,d\omega$	$e^{j\omega n}\to H\left(e^{j\omega}\right)e^{j\omega n}$ $x[n]\to y[n]=\int_{\omega}?\,H(e^{j\omega})e^{j\omega n}\,d\omega$

Eigenvalues and Eigenvectors

- In linear algebra, $v \in \mathbb{R}^n$ is **eigenvector** of matrix A with associated **eigenvalue** λ, if

$$Av = \lambda v$$

- Suppose A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ associated to eigenvectors $v_1, \ldots, v_n$, respectively.

- $v_1, \ldots, v_n$ form a basis of $\mathbb{R}^n$, i.e., $\forall x \in \mathbb{R}^n$ we have the following representation

$$x = \sum_{k=1}^{n} a_k v_k$$

Eigenvalues and Eigenfunctions of LTI Systems

- **CT Exponential $x(t) = e^{st}$**

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)}\, d\tau = e^{st} \int h(\tau)\, e^{-s\tau} d\tau \triangleq H(s) e^{st}$$

  - e^{st} is **eigenfunction** of LTI system with associated **eigenvalue**

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau \qquad \textbf{(system function)}$$

- **DT Exponential $x[n] = z^n$**

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k} \triangleq H(z) z^n$$

  - z^n is **eigenfunction** of LTI system with associated **eigenvalue**

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k} \qquad \textbf{(system function)}$$

Linear Combination of Eigenfunctions

- Following **eigenfunction** property and **superposition** property of LTI systems, we obtain

$$x(t)=\sum_k a_k e^{s_k t} \;\rightarrow\; y(t)=\sum_k a_k H(s_k) e^{s_k t}$$

$$x[n]=\sum_k a_k z_k^n \;\rightarrow\; y[n]=\sum_k a_k H(z_k) z_k^n$$

- **Input vs. output**
  - linear combinations of the same exponentials
  - different coefficients $\{a_k\} \mapsto \{a_k H(s_k)\}/\{a_k H(z_k)\}$
- **Questions**
  - Which functions are linear combinations of exponentials?
  - How to find coefficients $\{a_k\}$?

Basic Signals in Fourier Analysis

- If we focuses on $e^{j\omega t}$ for CT and $e^{j\omega n}$ for DT, i.e.,

$$e^{st} \xrightarrow{Re[s]=0} e^{j\omega t}$$

$$z^{n} \xrightarrow{|z|=1} e^{j\omega n}$$

- **Fourier analysis**
  - Decompose the signal as a linear combination of basic signals: $e^{j\omega t}/e^{j\omega n}$
  - Find out the response of the signal based on the response of these basic signals $e^{j\omega t}/e^{j\omega n}$

Fourier Series of CT Periodic Signals

- Recall CT signal is **periodic** with period T if

$$x(t) = x(t+T), \forall t \in \mathbb{R}$$

  - fundamental period T: smallest positive period
  - fundamental frequency $\omega_0 = \frac{2\pi}{T}$

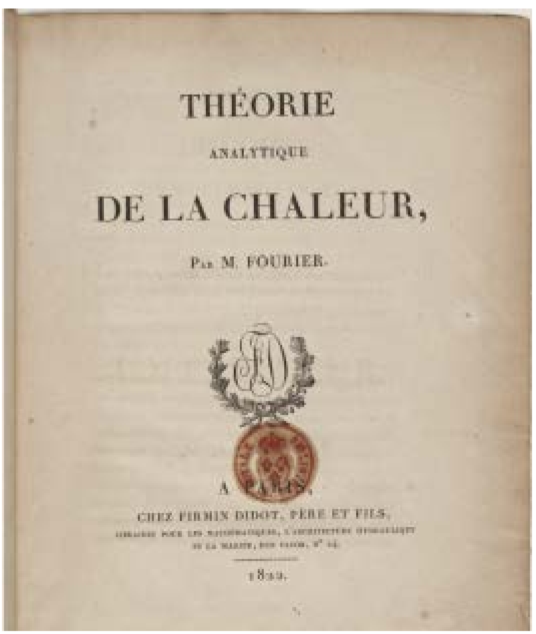
- **Fourier series** represent CT **periodic signals** in terms of **harmonically related** sinusoids or complex exponentials
  - $e^{j\omega_0 t}$ with fundamental period T and fundamental frequency ω_0
  - harmonically related: $\phi_k(t) = e^{jk\omega_0 t} = e^{jk2\pi t/T}$ periodic with fundamental frequency $k\omega_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$

$$x(t) = b_0 + \sum_{k=1}^{\infty} [b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)]$$

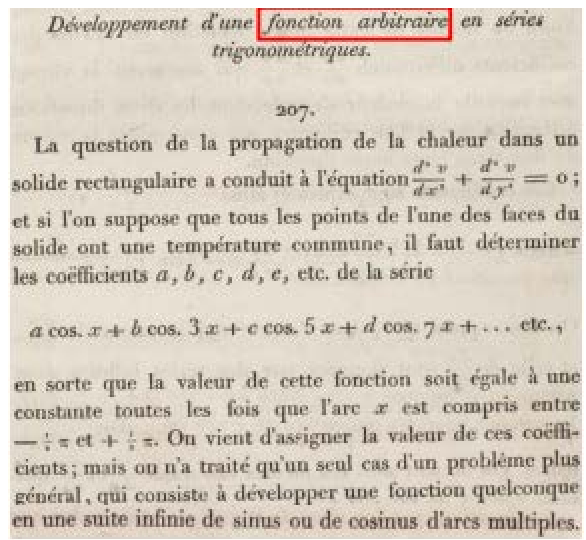
Joseph Fourier

- 1807, *Mémoire sur la propagation de la chaleur dans les corps solides (Memory on the propagation of heat in solid bodies)*
- *1822, Théorie analytique de la chaleur (The Analytical Theory of Heat)*


THÉORIE
ANALYTIQUE
DE LA CHALEUR,
Par M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,

1822.


Développement d'une fonction arbitraire en séries trigonométriques.

207.

La question de la propagation de la chaleur dans un solide rectangulaire a conduit à l'équation $\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} = 0$; et si l'on suppose que tous les points de l'une des faces du solide ont une température commune, il faut déterminer les coëfficients a, b, c, d, e, etc. de la série

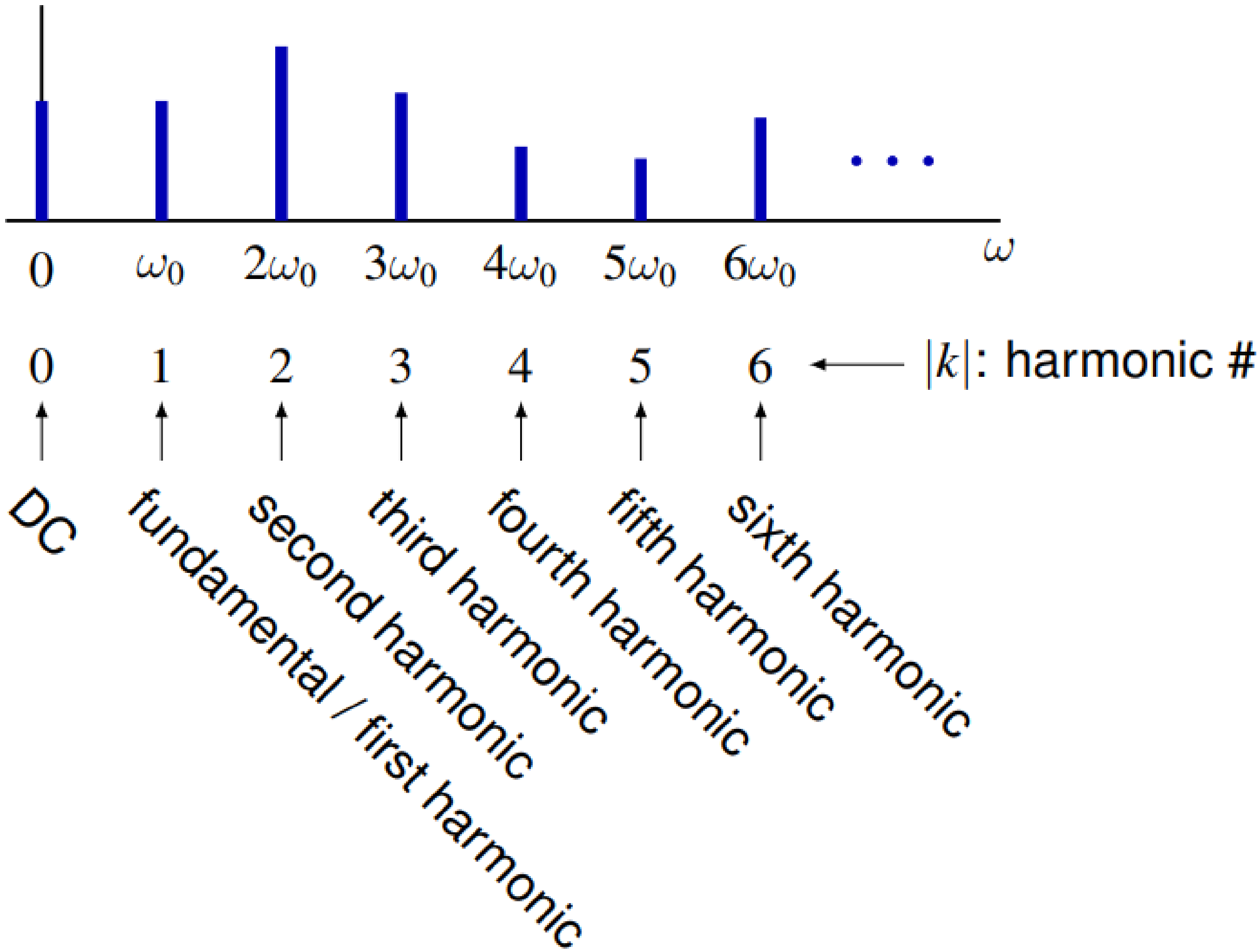
$$a \cos. x + b \cos. 3x + c \cos. 5x + d \cos. 7x + \dots \text{ etc.},$$

en sorte que la valeur de cette fonction soit égale à une constante toutes les fois que l'arc x est compris entre $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$. On vient d'assigner la valeur de ces coëfficients; mais on n'a traité qu'un seul cas d'un problème plus général, qui consiste à développer une fonction quelconque en une suite infinie de sinus ou de cosinus d'arcs multiples.

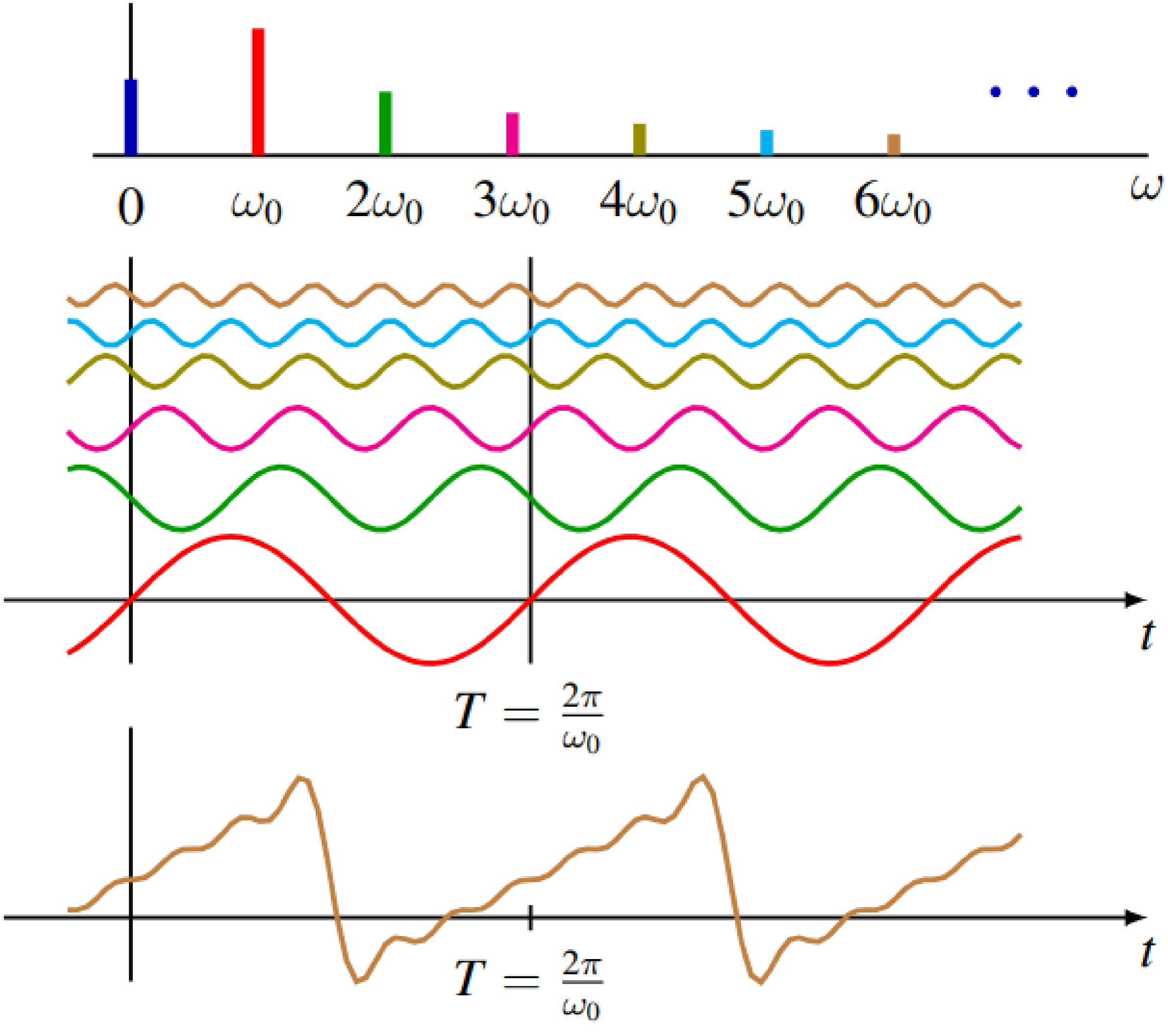
(https://gallica.bnf.fr/)

Jean-Baptiste Joseph Fourier (from Wikipedia)

Harmonics


0
ω_0
$2\omega_0$
$3\omega_0$
$4\omega_0$
$5\omega_0$
$6\omega_0$
ω
0 1 2 3 4 5 6
$|k|$: harmonic #
DC
fundamental / first harmonic
second harmonic
third harmonic
fourth harmonic
fifth harmonic
sixth harmonic

Harmonics


0
ω_0
$2\omega_0$
$3\omega_0$
$4\omega_0$
$5\omega_0$
$6\omega_0$
ω
t
$T = \frac{2\pi}{\omega_0}$
t
$T = \frac{2\pi}{\omega_0}$

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Orthonormality of Harmonics

- Recall for $k \neq 0$,

$$\int_a^b e^{jk\omega_0 t} dt = \frac{e^{jk\omega_0 b} - e^{jk\omega_0 a}}{jk\omega_0}$$

- Since $e^{jk\omega_0 t}$ has period T, then over any period

$$\frac{1}{T}\int_T e^{jk\omega_0 t} dt = \frac{1}{T}\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt = \delta[k]$$

- Define **inner product** between two signals with period T by

$$\langle f, g\rangle = \frac{1}{T}\int_T f(t)\overline{g(t)} dt$$

- $\{e^{jk\omega_0 t} \colon \forall k \in \mathbb{Z}\}$ is **orthonormal** system of functions

$$\langle e^{jm\omega_0 t}, e^{jn\omega_0 t}\rangle = \frac{1}{T}\int_T e^{j(m-n)\omega_0 t} dt = \delta[m-n], \forall m, n \in \mathbb{Z}$$

Orthogonality of Harmonics

- For Sinusoids, $\forall m, n \in \mathbb{Z}$,

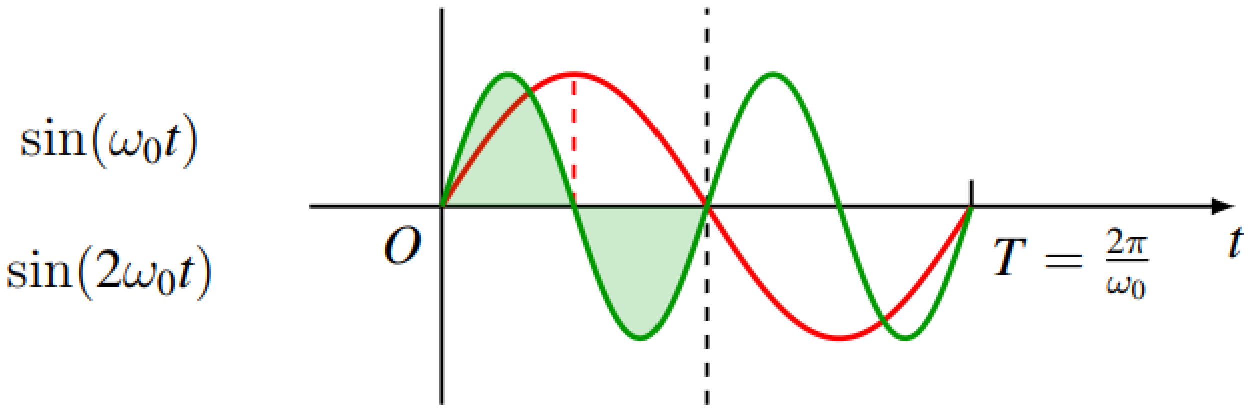
$$\langle \sin(m\omega_0 t), \sin(n\omega_0 t)\rangle = \frac{1}{2}\delta[m-n] - \frac{1}{2}\delta[m+n]$$

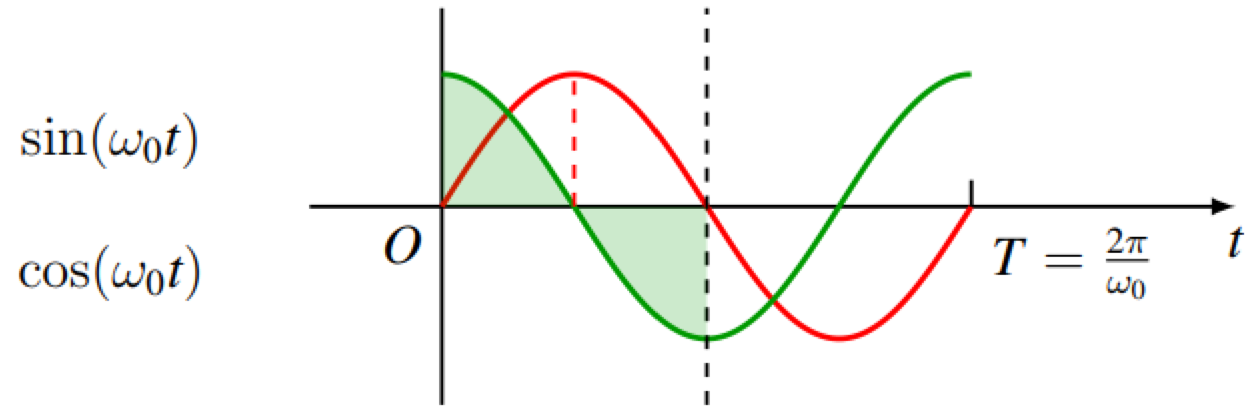
$$\langle \cos(m\omega_0 t), \cos(n\omega_0 t)\rangle = \frac{1}{2}\delta[m-n] + \frac{1}{2}\delta[m+n]$$

$$\langle \sin(m\omega_0 t), \cos(n\omega_0 t)\rangle = 0$$

- **Proof:**
  - Use Euler's relation and the orthonormality of $\{e^{jk\omega_0 t}: \forall k \in \mathbb{Z}\}$

Orthogonality of Harmonics





Complex Fourier Coefficients

- **Fourier series representation** of a CT signal with period T

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \omega_0=\frac{2\pi}{T}$$

- Find **Fourier coefficients** using orthonormality of $\{e^{jk\omega_0 t}: \forall k\in\mathbb{Z}\}$

$$\frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt=\langle x(t),e^{jn\omega_0 t}\rangle=\left\langle \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, e^{jn\omega_0 t}\right\rangle$$

$$=\sum_{k=-\infty}^{\infty} a_k \langle e^{jk\omega_0 t}, e^{jn\omega_0 t}\rangle$$

$$=\sum_{k=-\infty}^{\infty} a_k \delta[k-n]=a_n$$

Complex Fourier Coefficients

- **Synthesis equation**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

- ***N*-th partial sum**

$$S_N(x)(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- **Analysis equation**

$$a_k = \langle x(t), e^{jk\omega_0 t} \rangle = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t}\, dt$$

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Fourier Series Representation

- Fourier Series Representation of CT Periodic Signals
  - A periodic signal $x(t)$ can be represented as a linear combination of harmonically related complex exponentials (or sinusoids)
  - a_k: the **magnitude** and **phase** of ***k*-th harmonic component**
    - $a_0 = \frac{1}{T}\int_T x(t)dt$ - DC component
  - $\{a_k\}$: **Fourier series coefficients**, or spectrum of $x(t)$
    - $\{|a_k|\}$ - magnitude spectrum
    - $\{\sphericalangle a_k\}$ - phase spectrum

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Trigonometric Fourier Series

- For Sinusoids, $\forall m, n \in \mathbb{Z}$,

$$\langle \sin(m\omega_0 t), \sin(n\omega_0 t)\rangle = \frac{1}{2}\delta[m-n] - \frac{1}{2}\delta[m+n]$$

$$\langle \cos(m\omega_0 t), \cos(n\omega_0 t)\rangle = \frac{1}{2}\delta[m-n] + \frac{1}{2}\delta[m+n]$$

$$\langle \sin(m\omega_0 t), \cos(n\omega_0 t)\rangle = 0$$

- **Synthesis equation**

$$x(t) = b_0 + \sum_{k=1}^{\infty} [b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)]$$

- **Analysis equation**

$$b_k = \frac{2-\delta[k]}{T}\int_T x(t)\cos(k\omega_0 t)dt, \qquad c_k = \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)dt$$

Equivalence of Two Forms

- **Complex form**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- **Trigonometric form**

$$x(t) = b_0 + \sum_{k=1}^{\infty} [b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)]$$

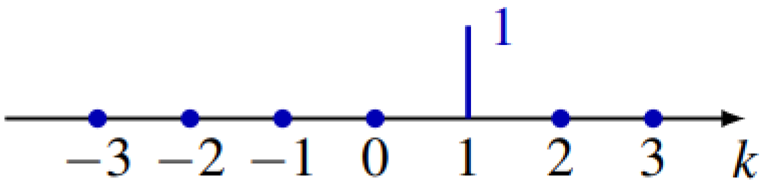
- **Conversion of coefficients** (by Euler's formula)

$$\begin{cases} b_0 = a_0 \\ b_k = a_k + a_{-k}, & k \geq 1 \\ c_k = j(a_k - a_{-k}), & k \geq 1 \end{cases} \qquad \begin{cases} a_0 = b_0 \\ a_k = 1/2(b_k - jc_k), & k \geq 1 \\ a_k = 1/2(b_k + jc_{-k}), & k \leq -1 \end{cases}$$

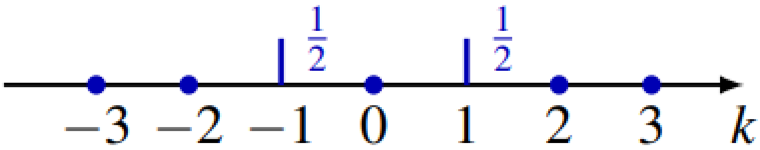
- **Note:** Negative frequencies in complex form introduced for mathematical convenience, no physical significance

Spectrum of $x(t)$

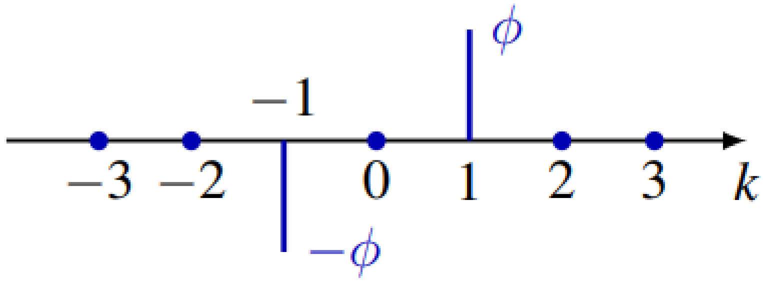
- **Example:**
  - $x(t) = \ e^{j\omega_0 t} \ \Longrightarrow \ a_k = \delta[k-1]$



- **Example:**
  - $x(t) = \cos(\omega_0 t + \phi) = \frac{e^{j\phi}}{2} e^{j\omega_0 t} + \frac{e^{-j\phi}}{2} e^{-j\omega_0 t}$
  - $a_1 = \frac{1}{2} e^{j\phi}, a_{-1} = \frac{1}{2} e^{-j\phi}, a_k = 0, \forall k \neq \pm 1$
  - **Magnitude spectrum** $|a_k|$

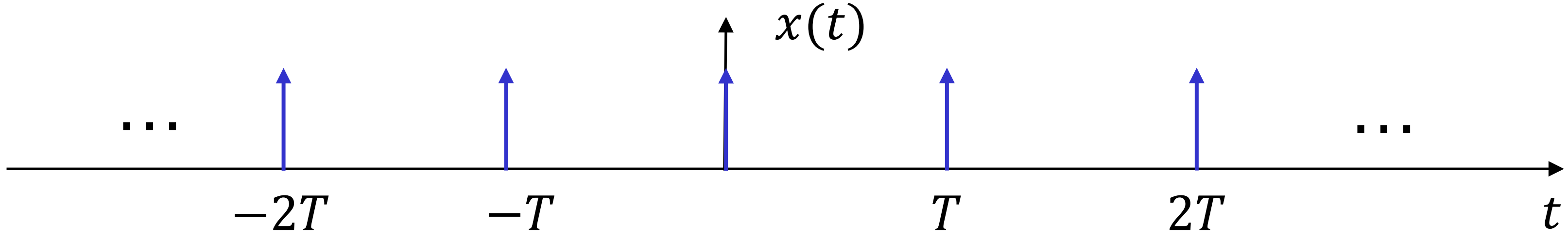


  - **Phase spectrum** $\measuredangle a_k$



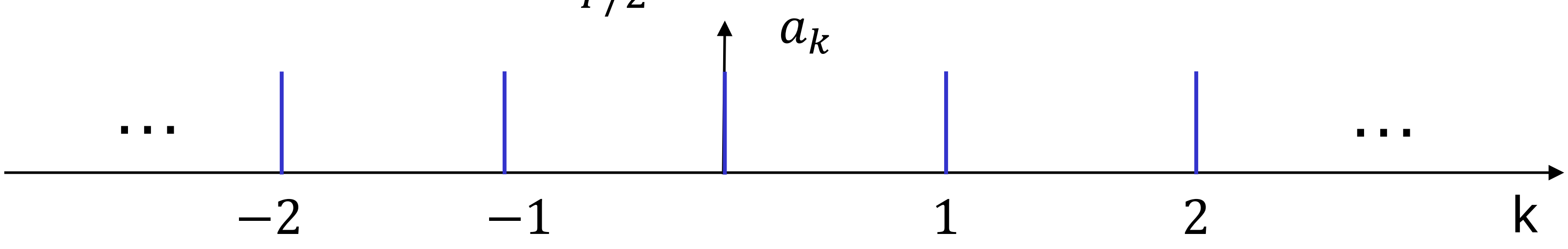
Example: Impulse Train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



- **Fourier coefficients**

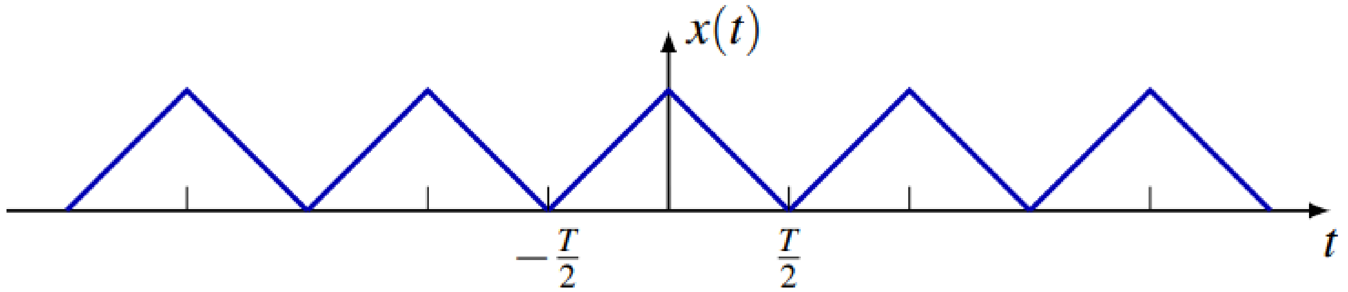
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi/Tt} dt = \frac{1}{T}$$



Example: Triangle Wave

- In one period

$$x(t) = 1 - \frac{2|t|}{T}, \qquad |t| \le \frac{T}{2}$$



- **Fourier coefficients**

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} 1 - \frac{2|t|}{T} dt = \frac{1}{2}, \qquad k = 0$$

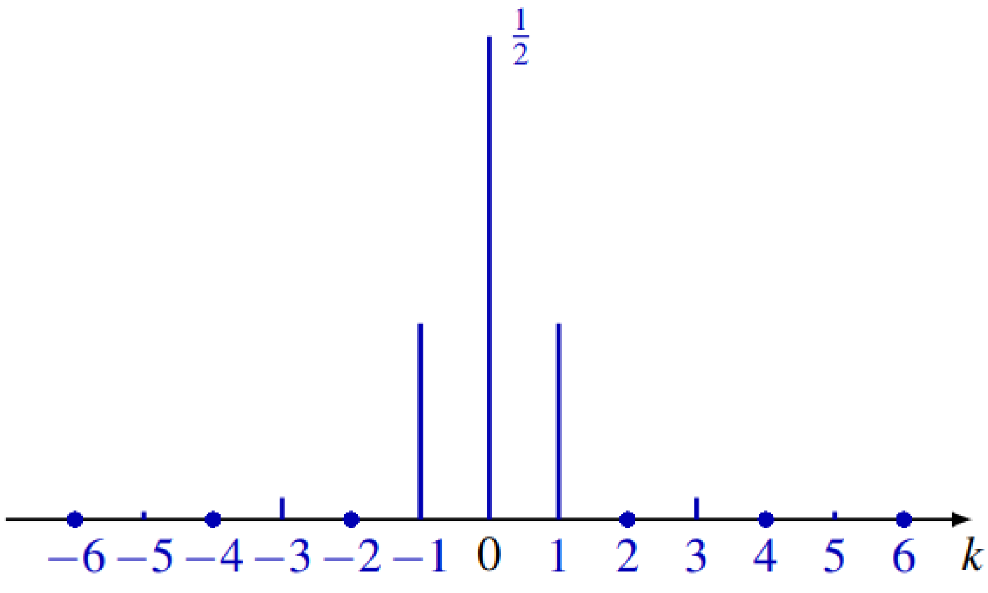
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \left(1 - \frac{2|t|}{T}\right) e^{-jk\omega_0 t} dt = \frac{2}{\pi^2 k^2}, k \neq 0, k \text{ odd}$$

$$a_k = 0, k \neq 0, k \text{ even}$$

Example: Triangle Wave

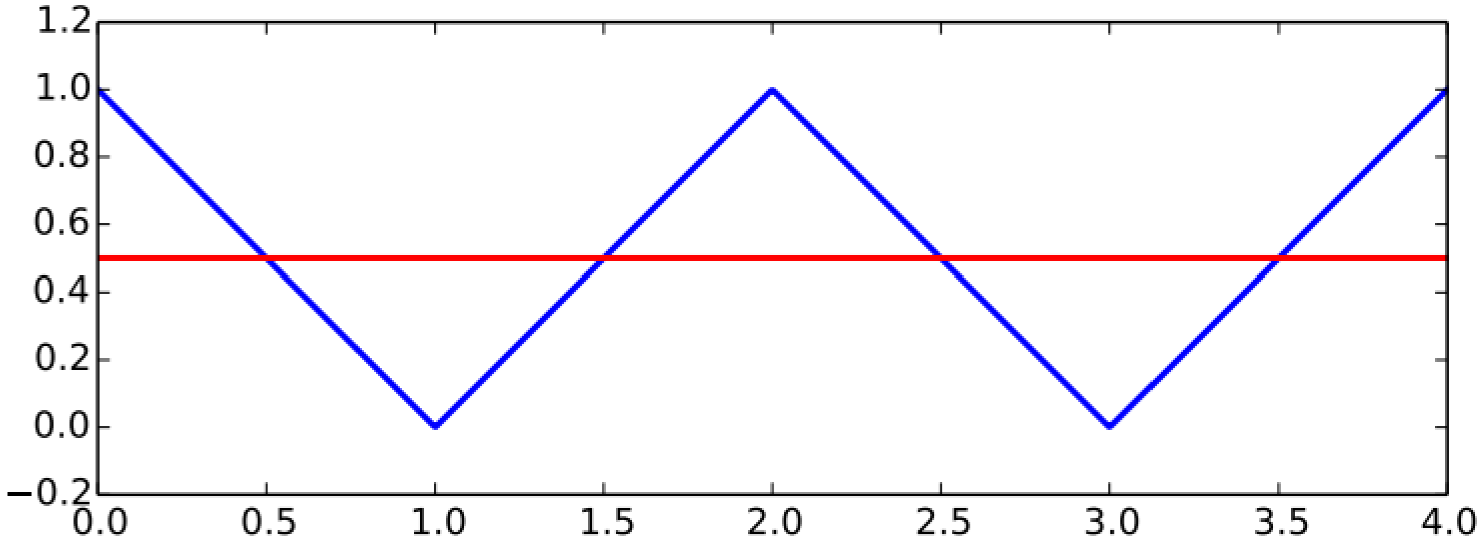
$$a_k = \begin{cases} 1/2, & k = 0 \\ \dfrac{2}{\pi^2 k^2} & k \neq 0 \text{ odd} \\ 0, & k \neq 0 \text{ even} \end{cases}$$

- **Spectrum** for fixed T, frequency spacing $\Delta\omega = \frac{2\pi}{T}$, $\omega_k = k\frac{2\pi}{T}$



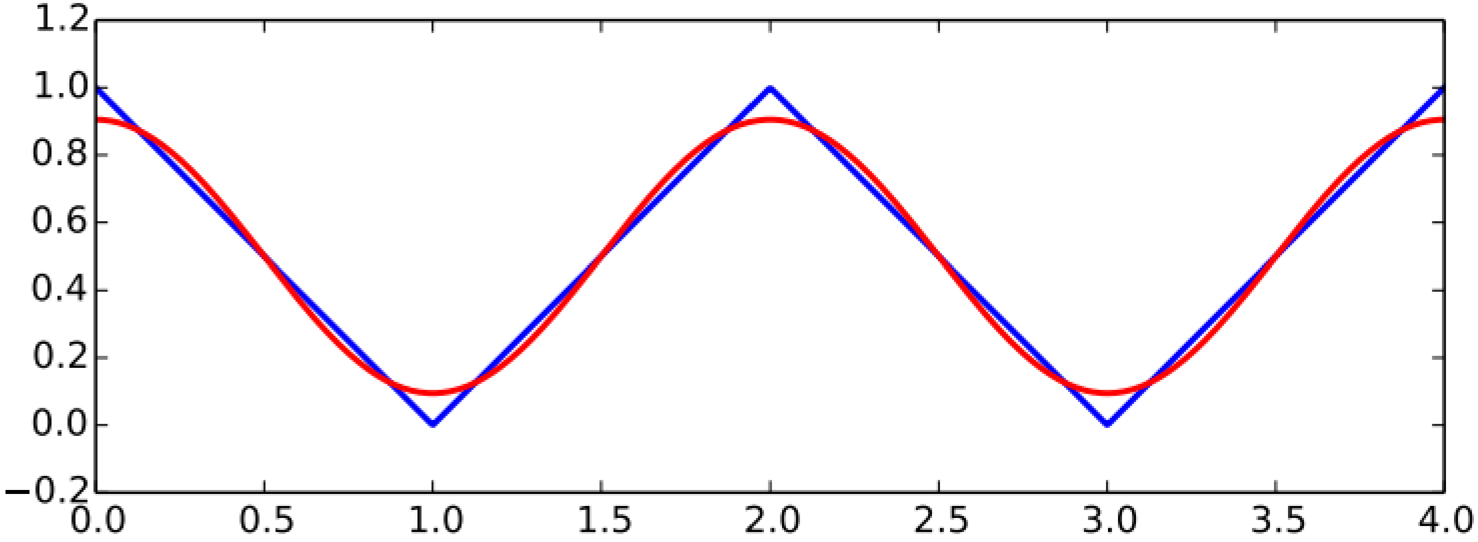
Example: Triangle Wave

$$S_0(x)(t) = \sum_{k=-0}^{0} a_k e^{jk\omega_0 t}$$



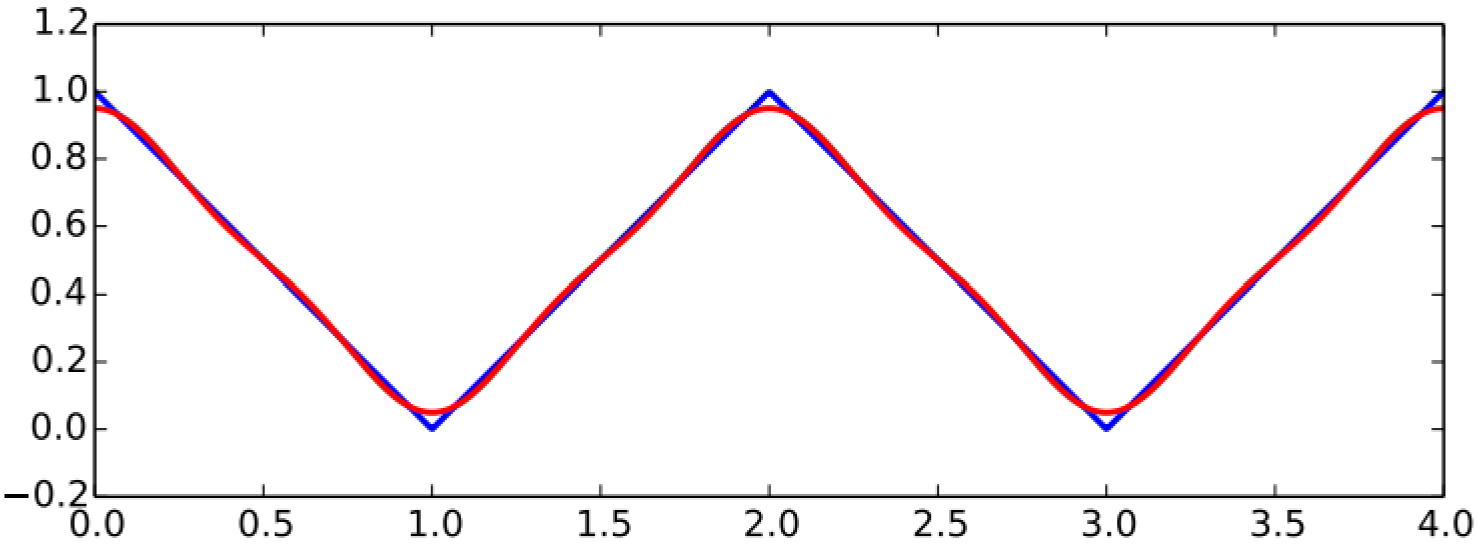
Example: Triangle Wave

$$S_1(x)(t) = \sum_{k=-1}^{1} a_k e^{jk\omega_0 t}$$



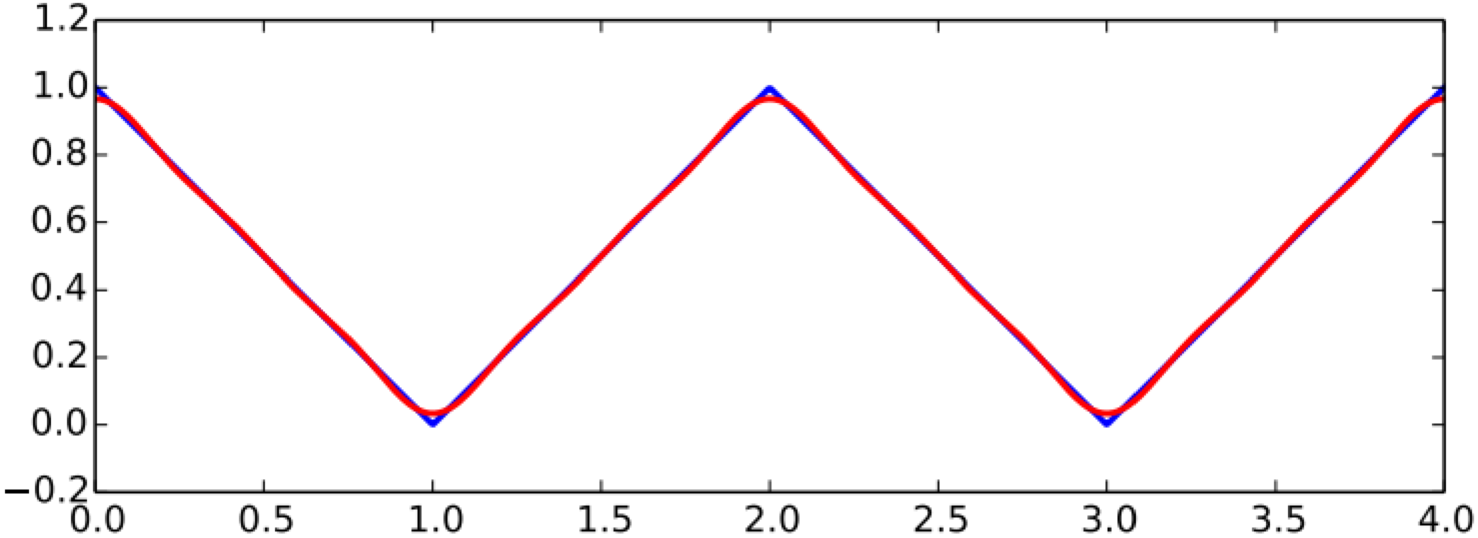
Example: Triangle Wave

$$S_3(x)(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$$



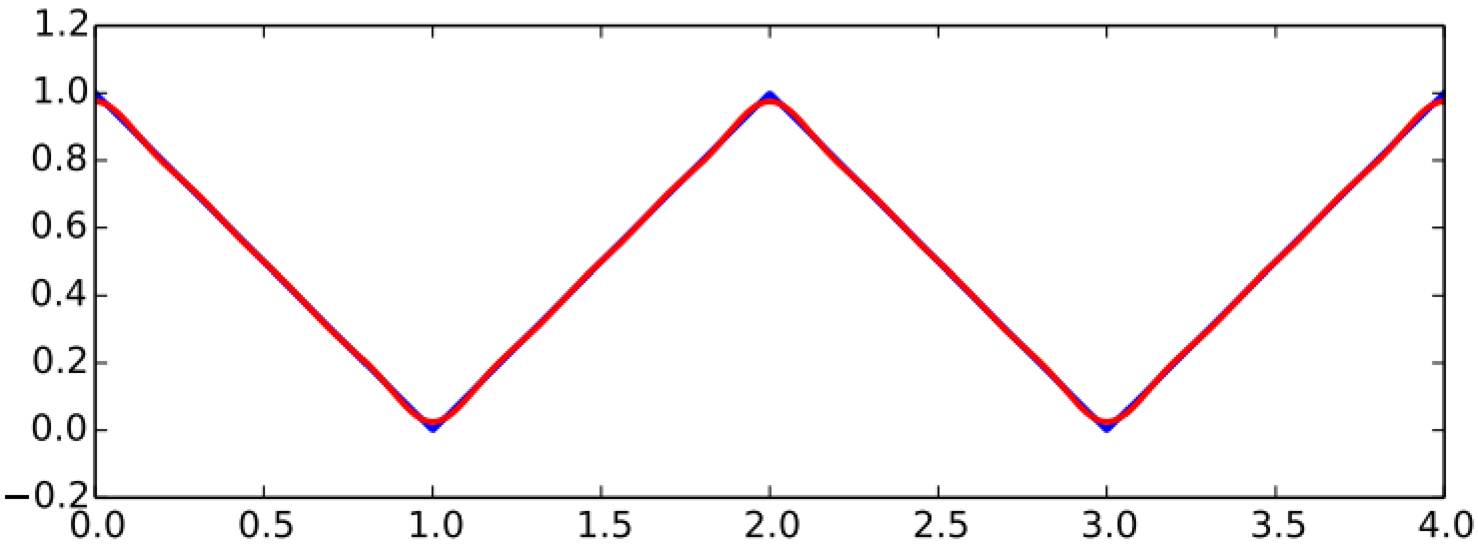
Example: Triangle Wave

$$S_5(x)(t) = \sum_{k=-5}^{5} a_k e^{jk\omega_0 t}$$



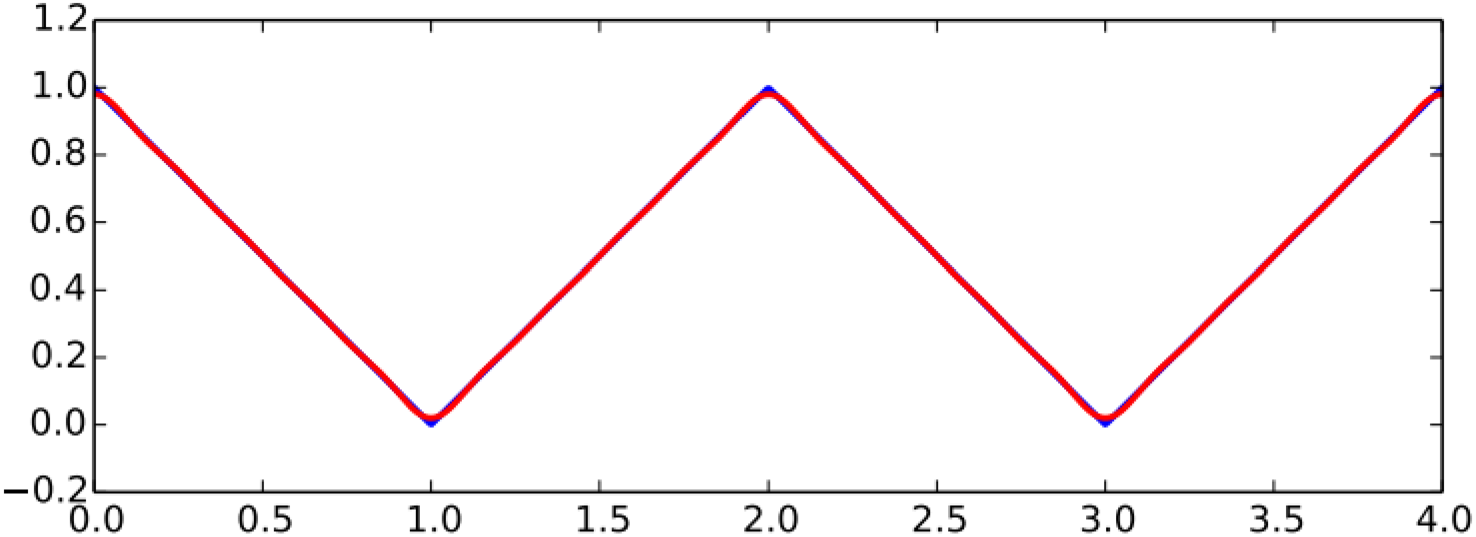
Example: Triangle Wave

$$S_7(x)(t) = \sum_{k=-7}^{7} a_k e^{jk\omega_0 t}$$



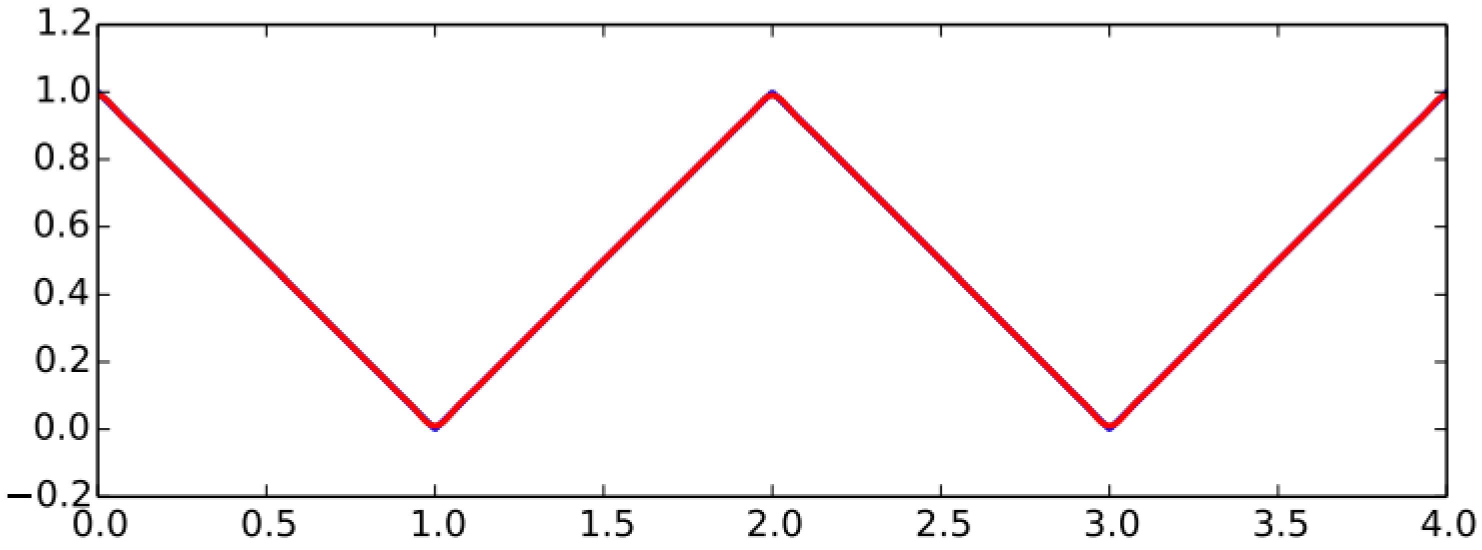
Example: Triangle Wave

$$S_9(x)(t) = \sum_{k=-9}^{9} a_k e^{jk\omega_0 t}$$



Example: Triangle Wave

$$S_{19}(x)(t) = \sum_{k=-19}^{19} a_k e^{jk\omega_0 t}$$



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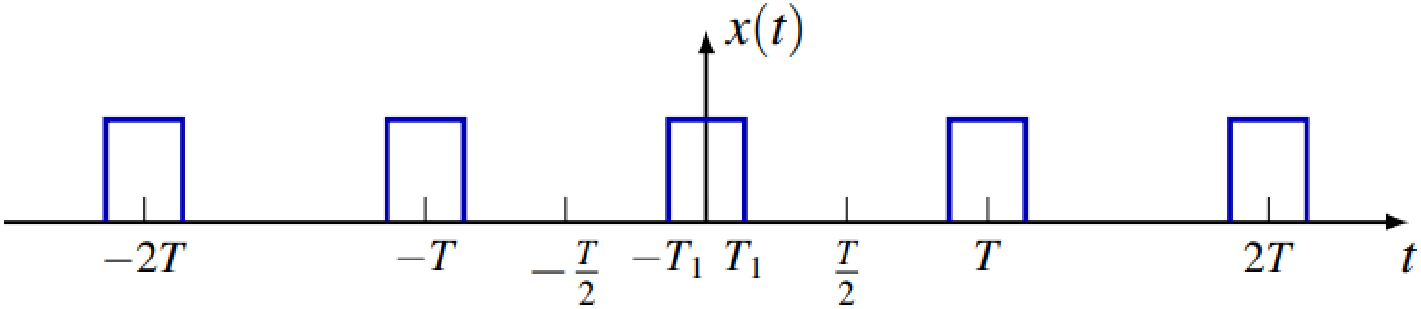
Example: Periodic Square Wave

- In one period

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



- **Fourier coefficients**

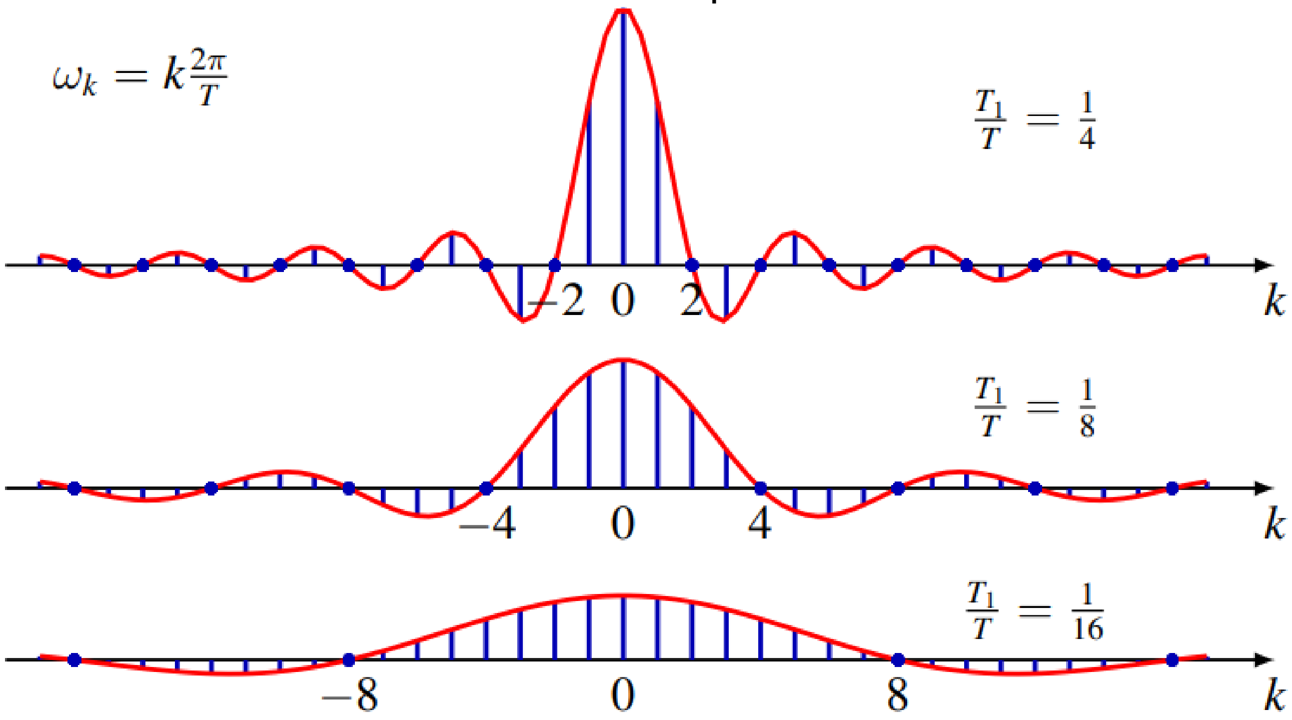
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}, k = 0$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

Example: Periodic Square Wave

$$a_k = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi} = \frac{2\sin(\omega_k T_1)}{\omega_k T}$$

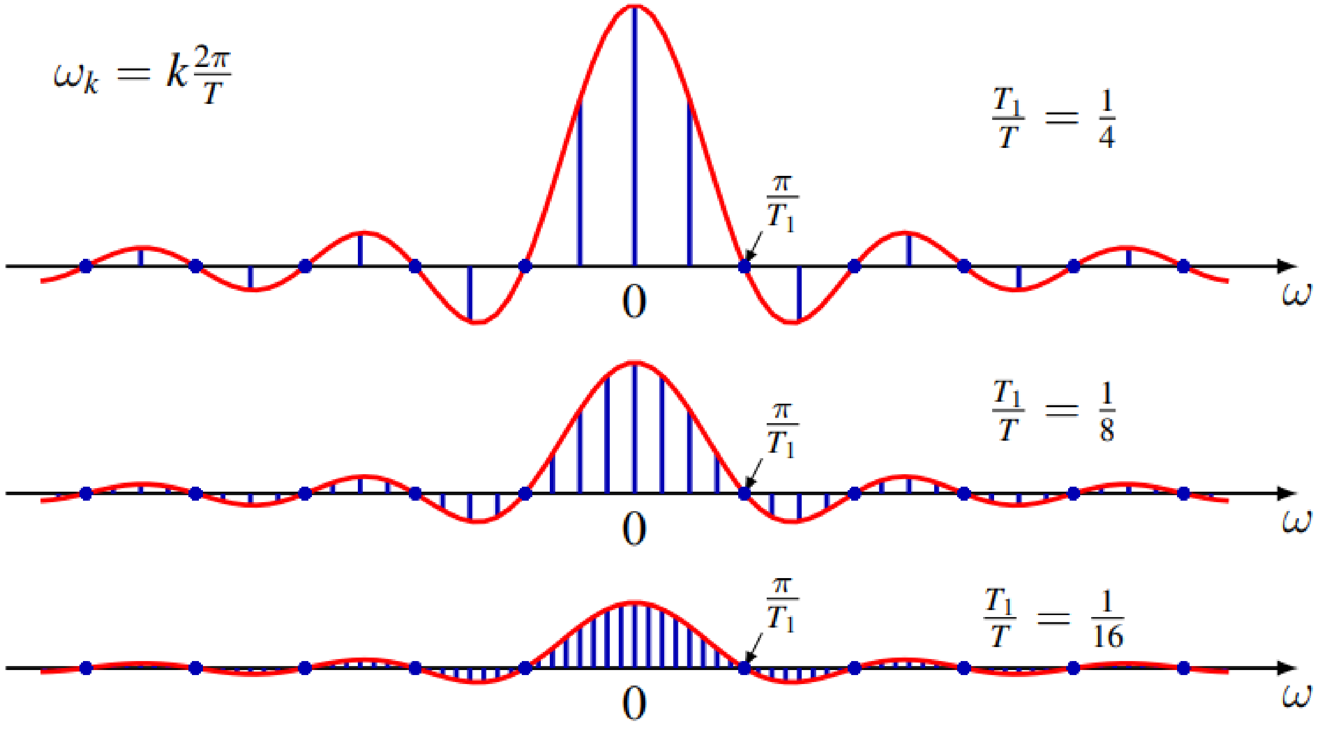
- **Spectra** for fixed T and different T_1



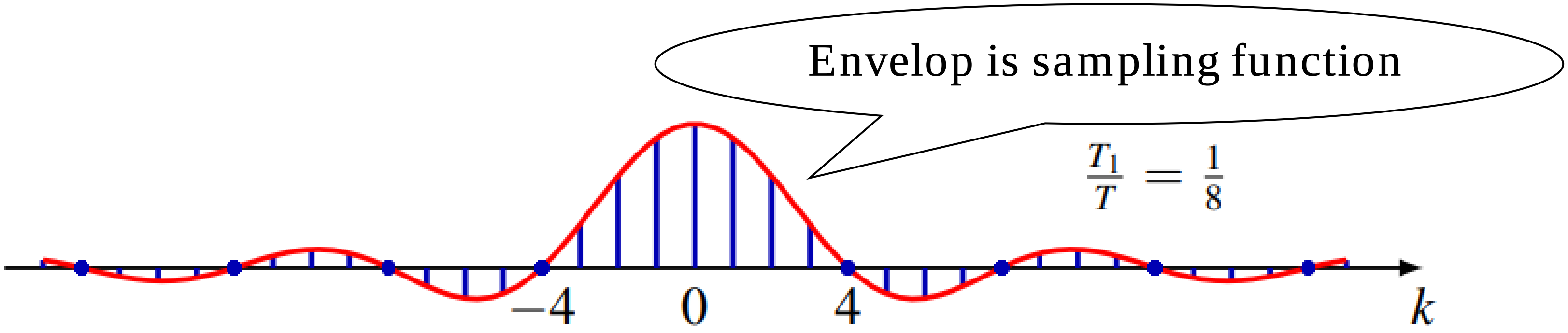
Example: Periodic Square Wave

$$a_k = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi} = \frac{2\sin(\omega_k T_1)}{\omega_k T}$$

- **Spectra** for fixed T_1 and different T



Example: Periodic Square Wave



- **Fourier coefficients**

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = 2\frac{T_1}{T}\mathrm{Sa}(k\omega_0 T_1)$$

  - where the **sampling function** is defined as

$$\mathrm{Sa}(x) = \frac{\sin(x)}{x}$$

  - first zero value at the k-th point satisfying $k\omega_0 T_1 = \pi$
  - main lobe of the signal is $T/2T_1$(Hz), which is the bandwidth of the signal

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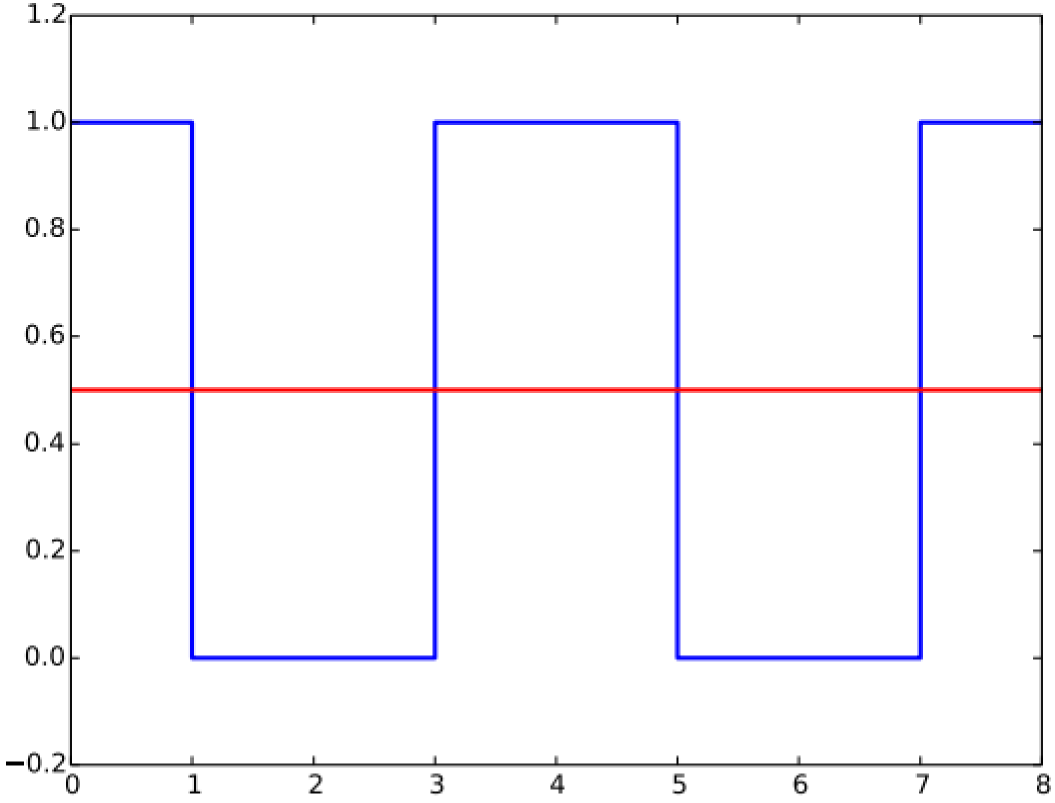
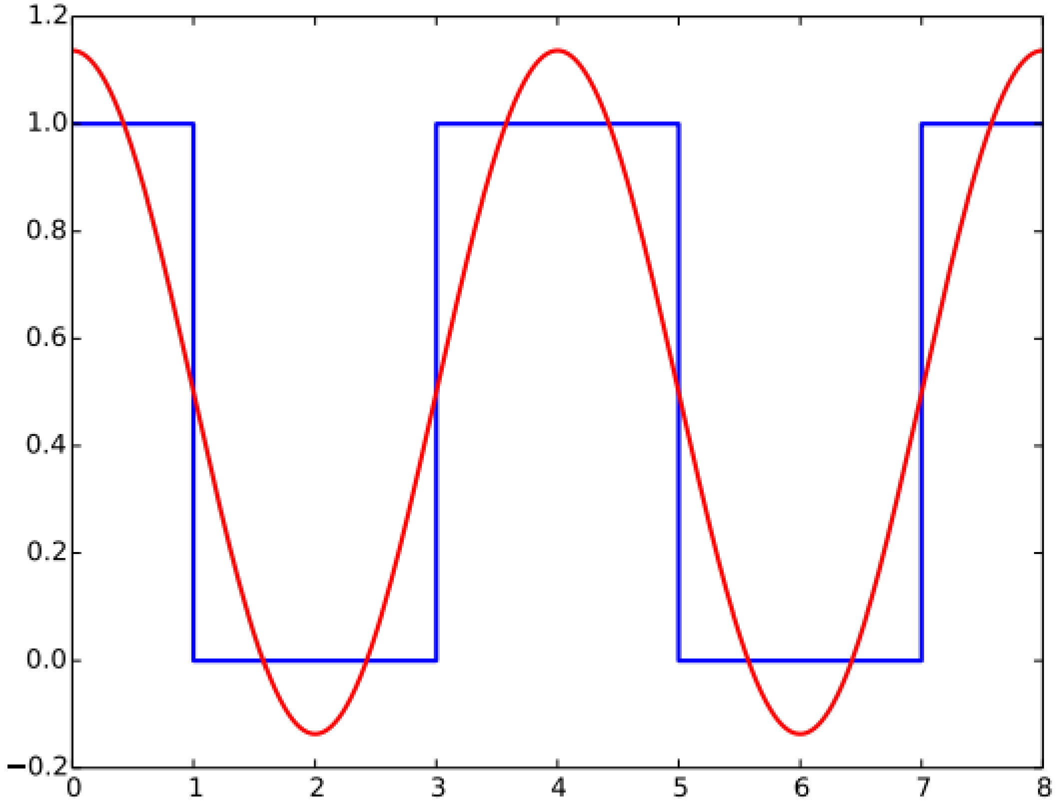
Example: Periodic Square Wave

$$S_0(x)(t) = \sum_{k=-0}^{0} a_k e^{jk\omega_0 t}$$



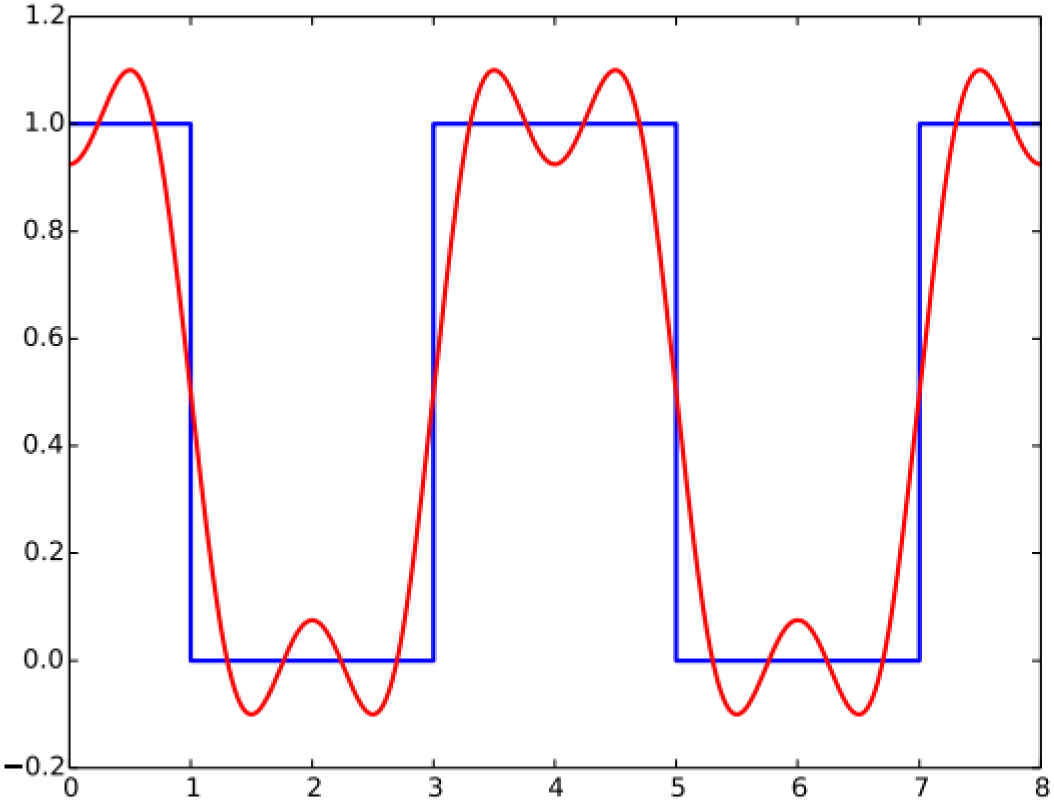
Example: Periodic Square Wave

$$S_1(x)(t) = \sum_{k=-1}^{1} a_k e^{jk\omega_0 t}$$



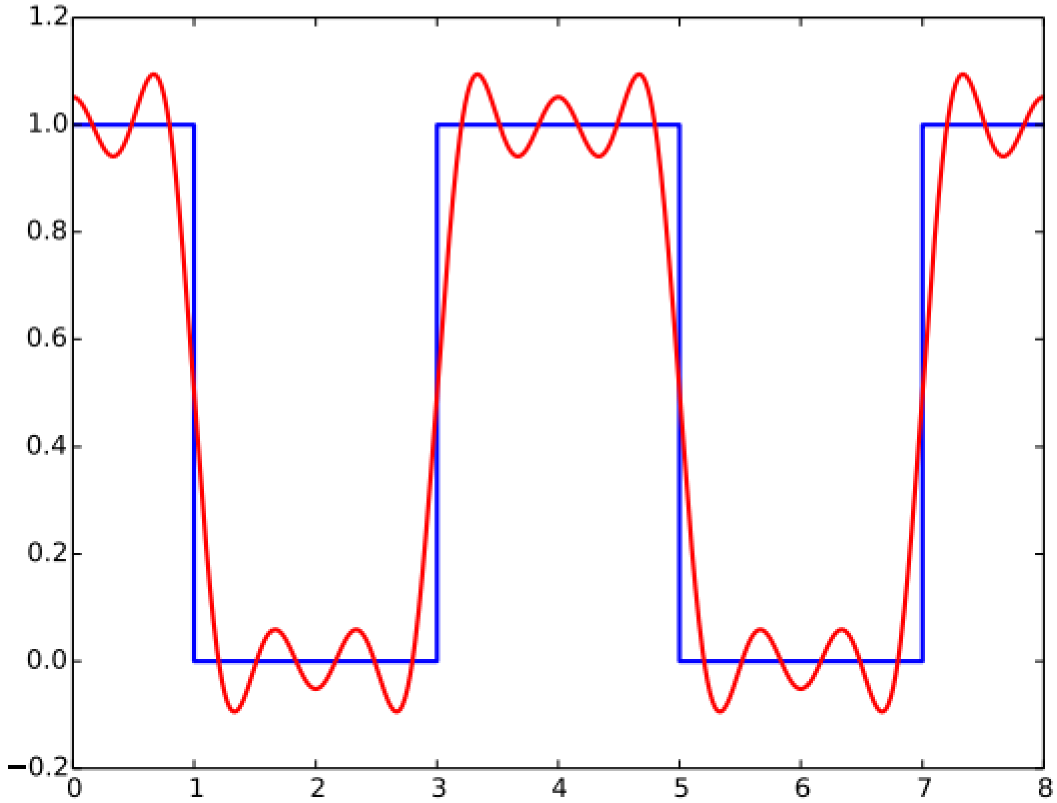
Example: Periodic Square Wave

$$S_3(x)(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$$



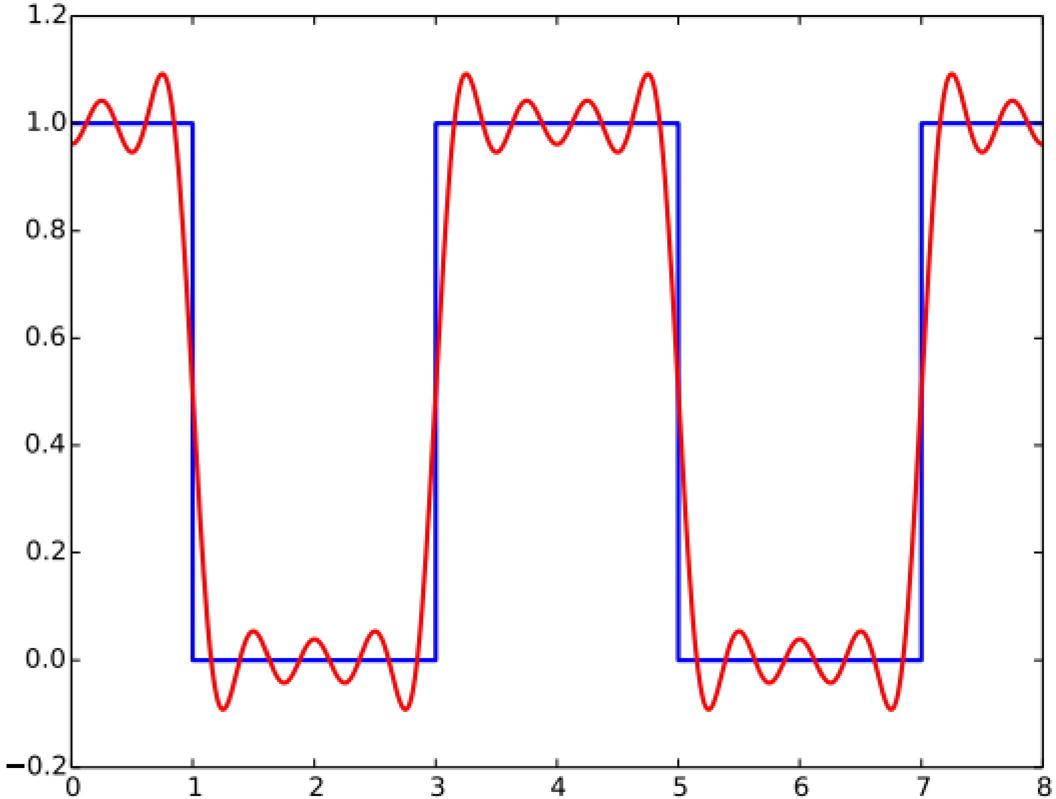
Example: Periodic Square Wave

$$S_5(x)(t) = \sum_{k=-5}^{5} a_k e^{jk\omega_0 t}$$



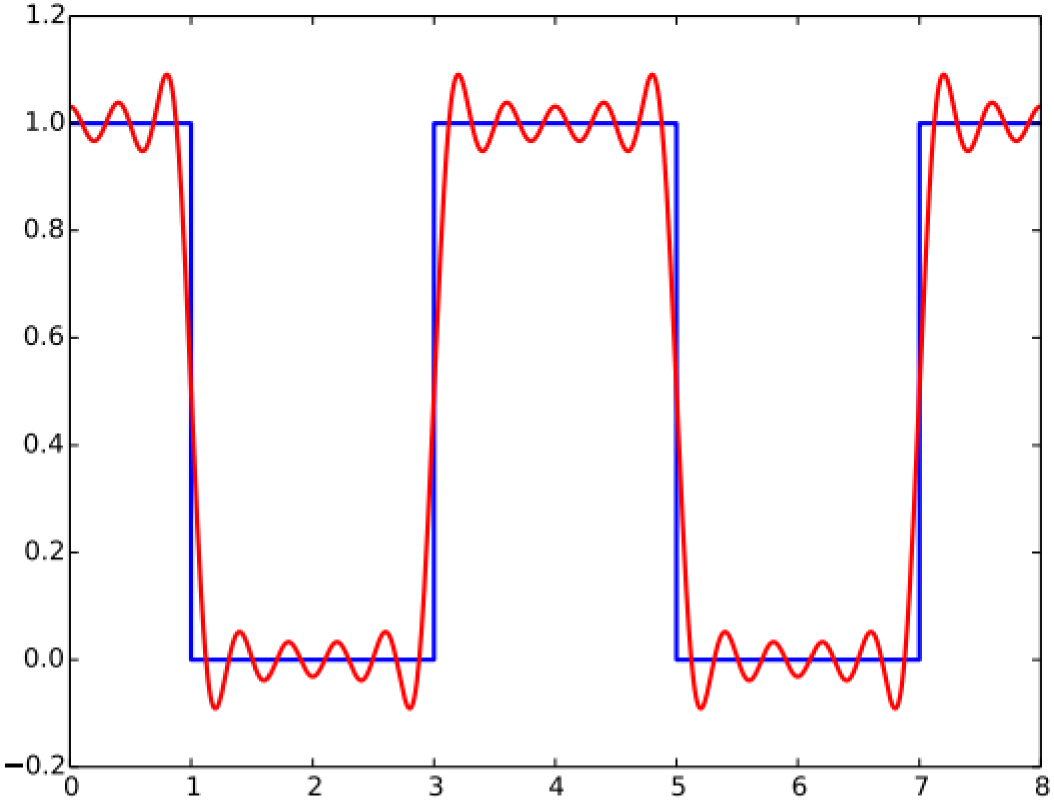
Example: Periodic Square Wave

$$S_7(x)(t) = \sum_{k=-7}^{7} a_k e^{jk\omega_0 t}$$



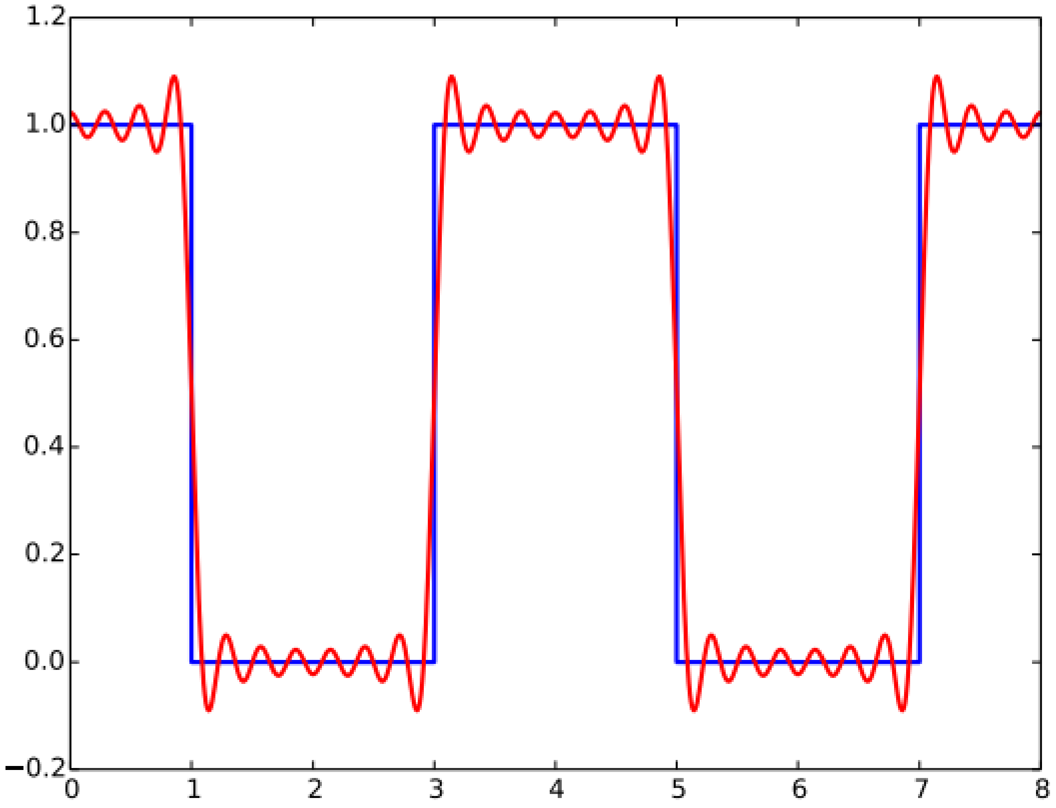
Example: Periodic Square Wave

$$S_9(x)(t) = \sum_{k=-9}^{9} a_k e^{jk\omega_0 t}$$



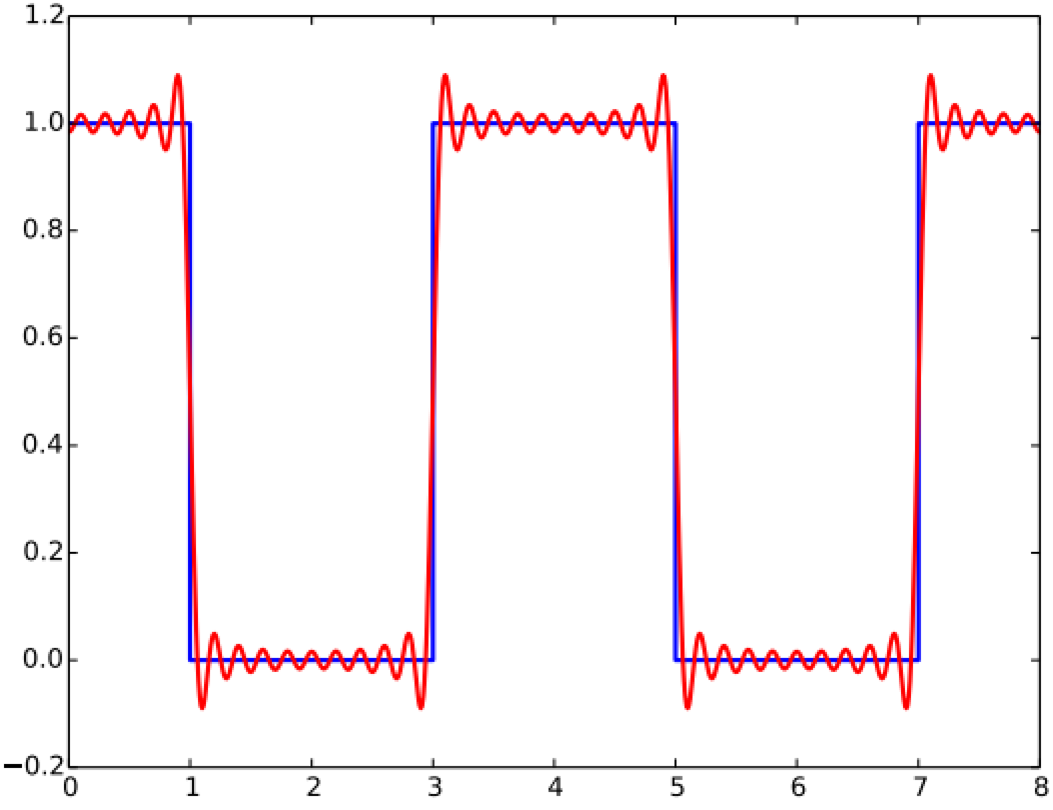
Example: Periodic Square Wave

$$S_{13}(x)(t) = \sum_{k=-13}^{13} a_k e^{jk\omega_0 t}$$



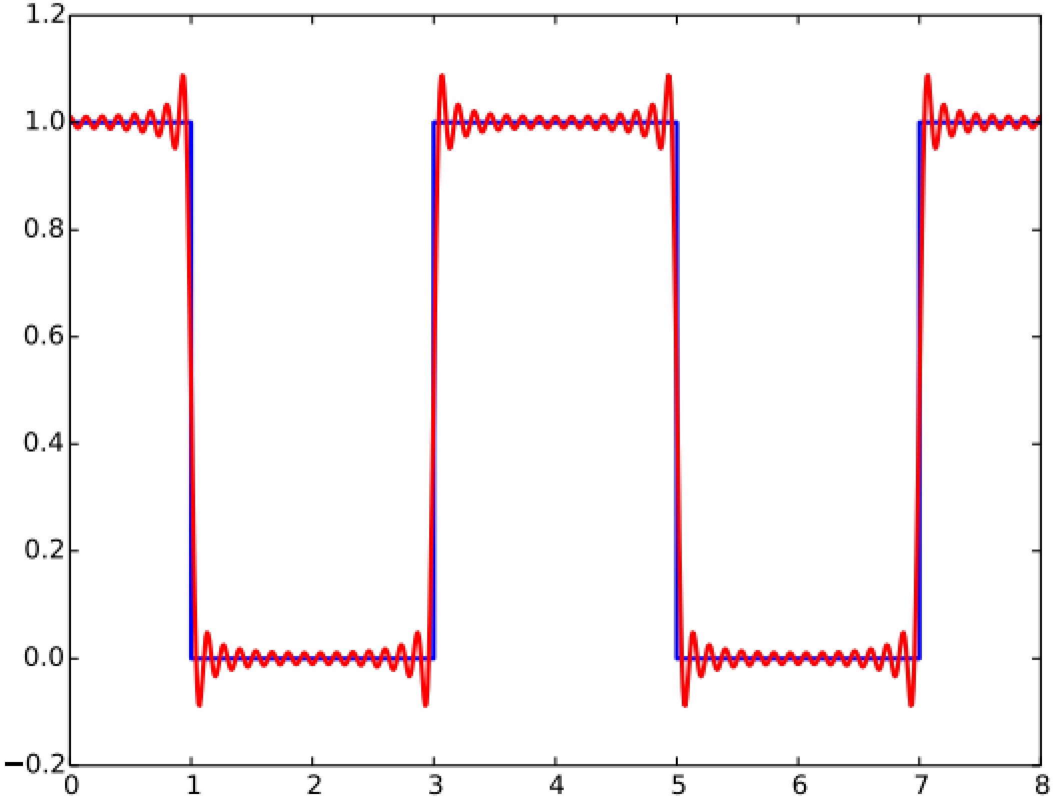
Example: Periodic Square Wave

$$S_{19}(x)(t) = \sum_{k=-19}^{19} a_k e^{jk\omega_0 t}$$



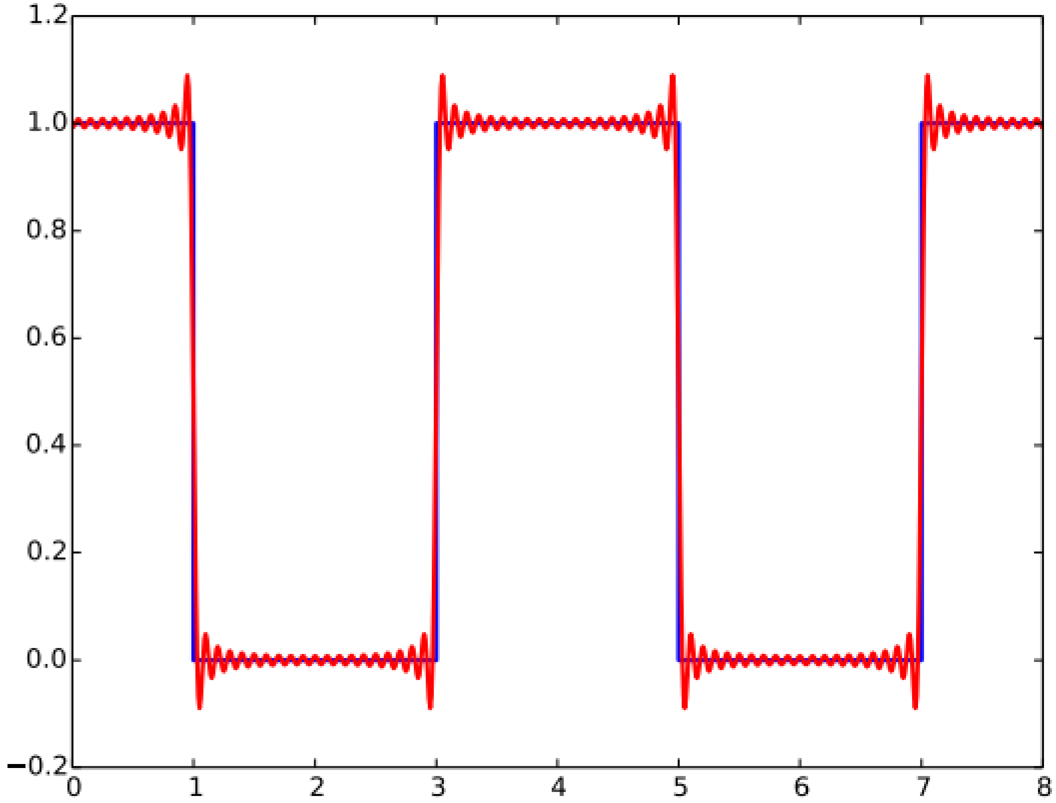
Example: Periodic Square Wave

$$S_{29}(x)(t) = \sum_{k=-29}^{29} a_k e^{jk\omega_0 t}$$



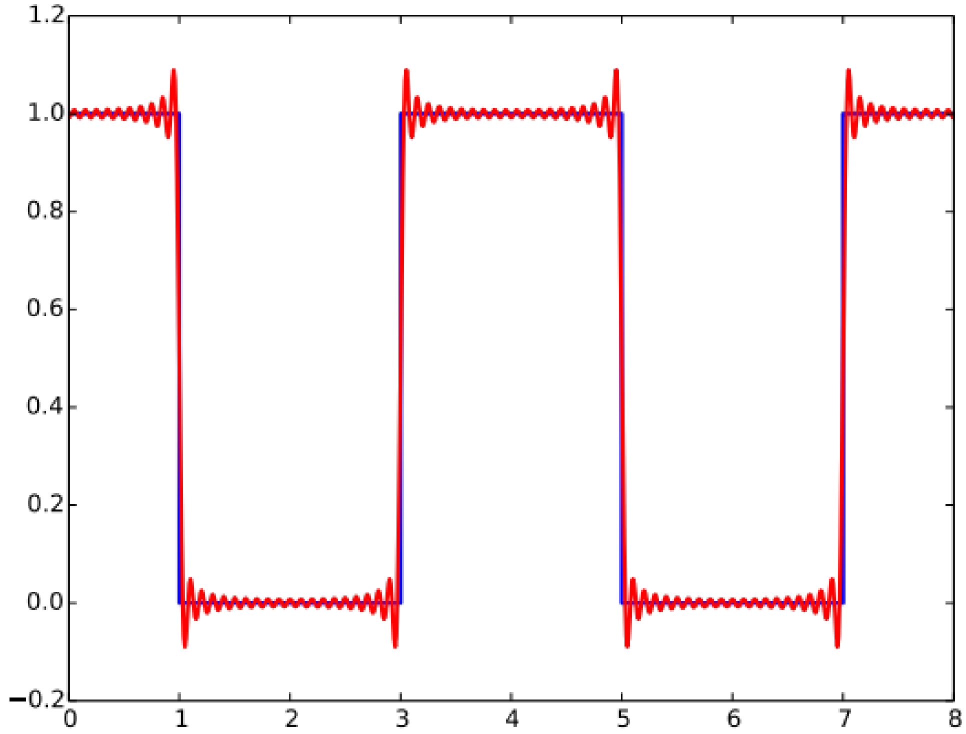
Example: Periodic Square Wave

$$S_{39}(x)(t) = \sum_{k=-39}^{39} a_k e^{jk\omega_0 t}$$



Gibbs Phenomenon

- Partial sums of Fourier series “ring” at jump discontinuity
  - overshoot gets closer and closer to discontinuity
  - overshoot approaches ≈ 9% of jump size



Convergence of Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt$$

- **Question:** can any CT periodic signal be represented by a Fourier series?
  - **No** in general, although it applies to a extremly large class of periodic signals
  - e.g., the integral in analysis equation (i.e., a_k) may diverge
  - e.g., even a_k is all finite, the synthesis equation may not converge to the original signal $x(t)$

Finite energy condition

- **Key point:** What do we mean by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} ?$$

- One useful notion for engineers:
  - **no energy** in **approximation error**

$$e(t) \triangleq x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}, \qquad \int_T |e(t)|^2 dt = 0$$

- **Finite energy condition**

$$\int_T |x(t)|^2 dt < \infty$$

  - **Note:** does not guarantee $x(t)$ equal to its Fourier representation at any time t, but guarantees no energy in approximation error

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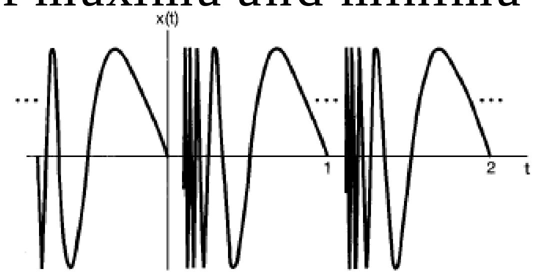
Dirichlet Conditions

- **Condition 1.**
  - $x(t)$ is absolutely integrable over one period, i.e.,

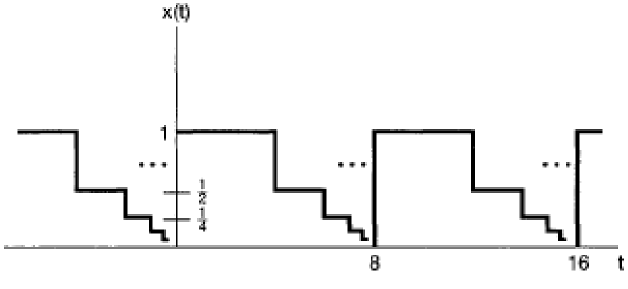
$$\int_T |x(t)|dt < \infty$$

- **Condition 2.**
  - During any period, $x(t)$ has a finite number of maxima and minima
  - **Example:** Condition 2 violated

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$



- **Condition 3.**
  - During any single period, $x(t)$ has only a finite number of discontinuities with finite values
  - **Example:** Condition 3 violated



Dirichlet Conditions

- Signals **violating Dirichlet conditions**
  - pathological in nature
  - do not typically arise in practice

- Signals **satisfying Dirichlet conditions**
  - 1) **no discontinuities**:
    - Fourier series = $x(t)$ at any time point t
  - 2) **isolated discontinuities**:
    - Fourier series = $x(t)$ at time points where $x(t)$ continuous
    - Fourier series = $1/2[x(t_{0-}) + x(t_{0+})]$ at isolated point t_0
    - Signal only differ at isolated discontinuity points

      $\Longrightarrow$ integrals of both signals over any interval are identical

      $\Longrightarrow$ behave identically under convolution

      $\Longrightarrow$ **identical for analysis of LTI systems**

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Fourier Series

- **Fourier series** of $x(t)$ with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Correspondence between periodic functions and doubly infinite sequences, **time domain** vs. **frequency domain**

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

  - a_k consists of expansion coefficients of $x(t)$ in "basis" $\{e^{jk\omega_0 t}\}$
  - a_k alone does **not** uniquely determine $x(t)$
  - also need to know basis functions, or equivalently, period T or fundamental frequency ω_0
  - same coefficients with different bases (periods) generate different functions

Linearity

- If $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$, $y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$ have **same** period T, so does their linear combination $Ax(t) + By(t)$, and

$$Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} Aa_k + Bb_k$$

- **Proof:**

$$\begin{aligned} Aa_k + Bb_k &= \langle Ax(t) + By(t), e^{jk\omega_0 t} \rangle \\ &= A\langle x(t), e^{jk\omega_0 t} \rangle + B\langle y(t), e^{jk\omega_0 t} \rangle = Aa_k + Bb_k \end{aligned}$$

- **Alternative proof:**

$$\left.\begin{aligned} x(t) &= \sum\nolimits_k a_k e^{jk\omega_0 t} \\ y(t) &= \sum\nolimits_k b_k e^{jk\omega_0 t} \end{aligned}\right\} \Longrightarrow Ax(t) + By(t) = \sum\nolimits_k (Aa_k + Bb_k) e^{jk\omega_0 t}$$

Time Shifting

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, and $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = e^{-jk\omega_0 t_0} a_k$$

Time shift ⇔ linear phase change in frequency

- **Proof:**

$$b_k = \frac{1}{T}\int_T x(t - t_0) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau$$

$$= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau = e^{-jk\omega_0 t_0} \cdot a_k$$

- **Example:**

$$\cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}, \sin(t) = \cos\left(t - \frac{\pi}{2}\right) = \frac{-j}{2}e^{jt} + \frac{j}{2}e^{-jt}$$

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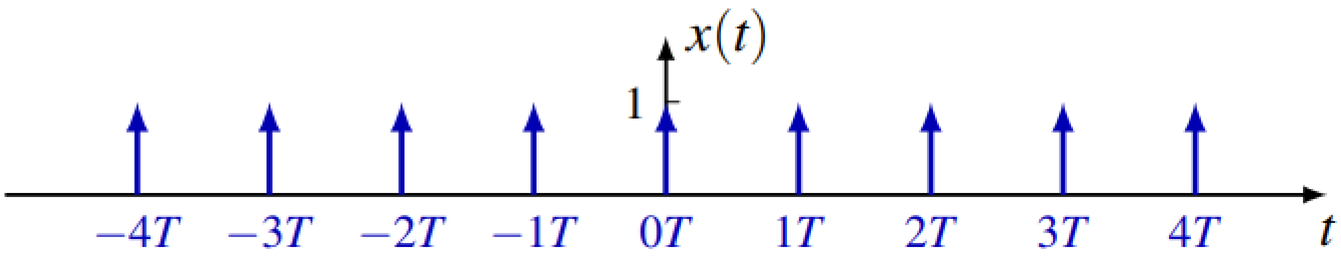
Example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



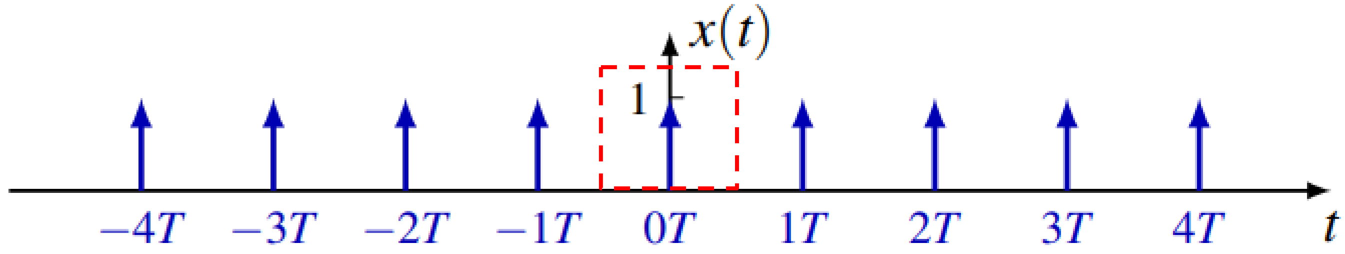
- For "good" enough function $\phi(t)$, e.g., infinitely differentiable function with compact support

$$\phi(t) * x(t) = \sum_{n=-\infty}^{\infty} \phi(t - nT)$$

$$\int_{\mathbb{R}} \phi(t)x(t)dt = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} \phi(t)\delta(t - nT)dt = \sum_{n=-\infty}^{\infty} \phi(nT)$$

Example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



- **Fourier coefficients and Fourier series**

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}, \forall k \in \mathbb{Z}$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t} = \lim_{N\to\infty} \frac{1}{T} D_N\left(\frac{2\pi}{T}t\right)$$

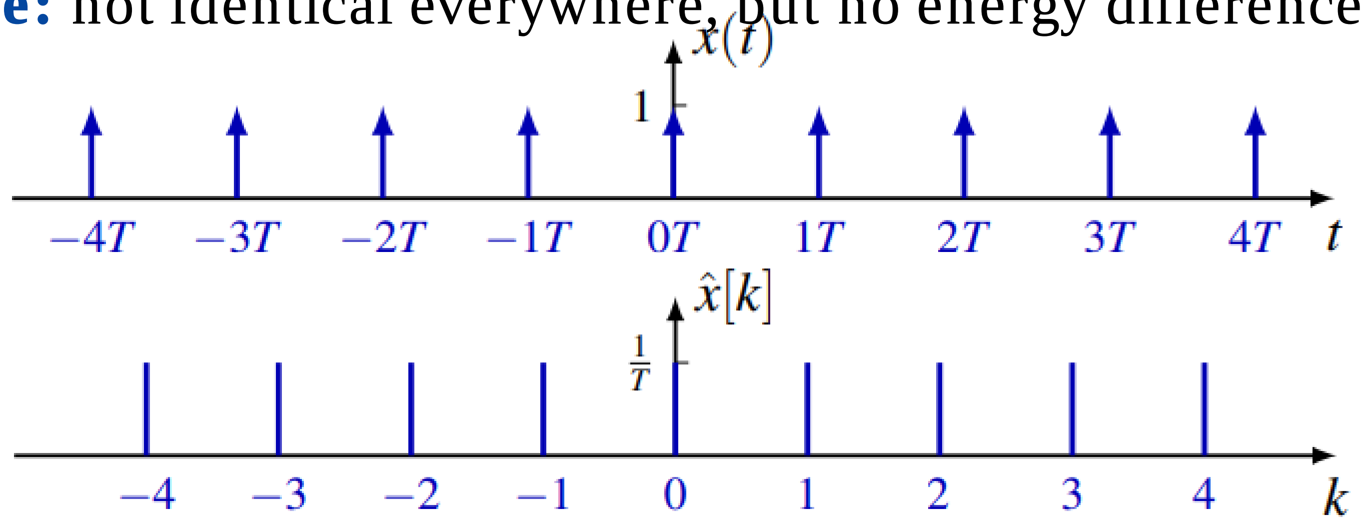
where D_N is Dirichlet kernel

Example: Periodic Impulse Train

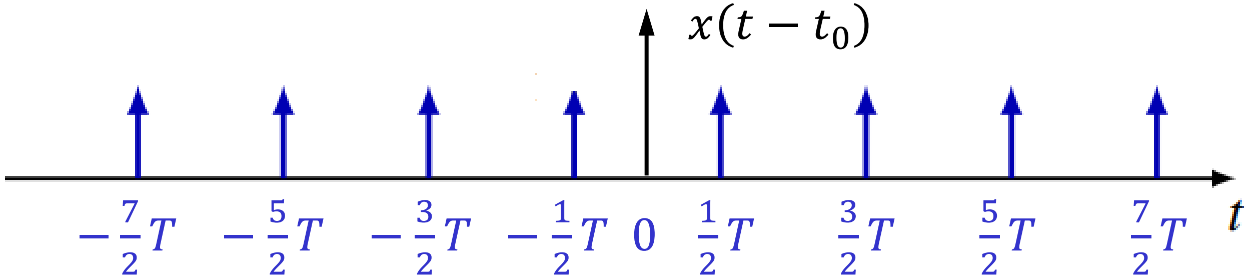
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

$$\int_T \left| x(t) - \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t} \right|^2 = 0$$

- **Note:** not identical everywhere, but no energy difference



Time Shifted by Half Period



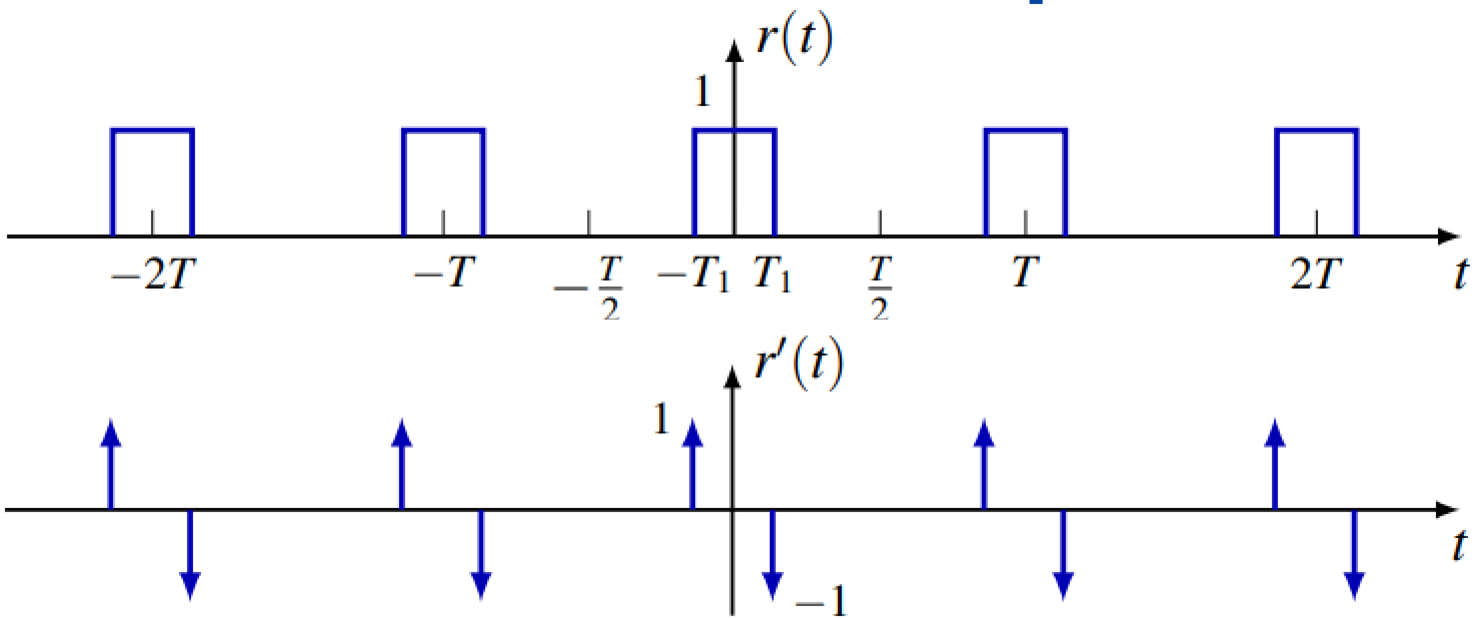
- **Fourier coefficients of the periodic impulse train $x(t)$**

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{T}$$

- **Fourier coefficients of $x(t)$ shifted by half period**

$$x\left(t-\frac{T}{2}\right) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = e^{-jk\frac{2\pi}{T}\cdot\frac{T}{2}} a_k = e^{-jk\pi}\cdot a_k = (-1)^k a_k$$

Relation with Periodic Square Wave



- **Recall** $r(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = \dfrac{\sin(k\omega_0 T_1)}{k\pi}, \forall k \neq 0$

$$r'(t) = x(t+T_1) - x(t-T_1)$$

$$r'(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = \left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)\frac{1}{T} = \frac{2j\sin(k\omega_0 T_1)}{T}, \forall k \neq 0$$

$$\Rightarrow c_k = \frac{1}{jk\omega_0} b_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \forall k \neq 0$$

Frequency Shifting

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, and $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$e^{jM\omega_0 t}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_{k-M}$$

Modulation by harmonic exponential ⟺ frequency shift

- **Proof:**

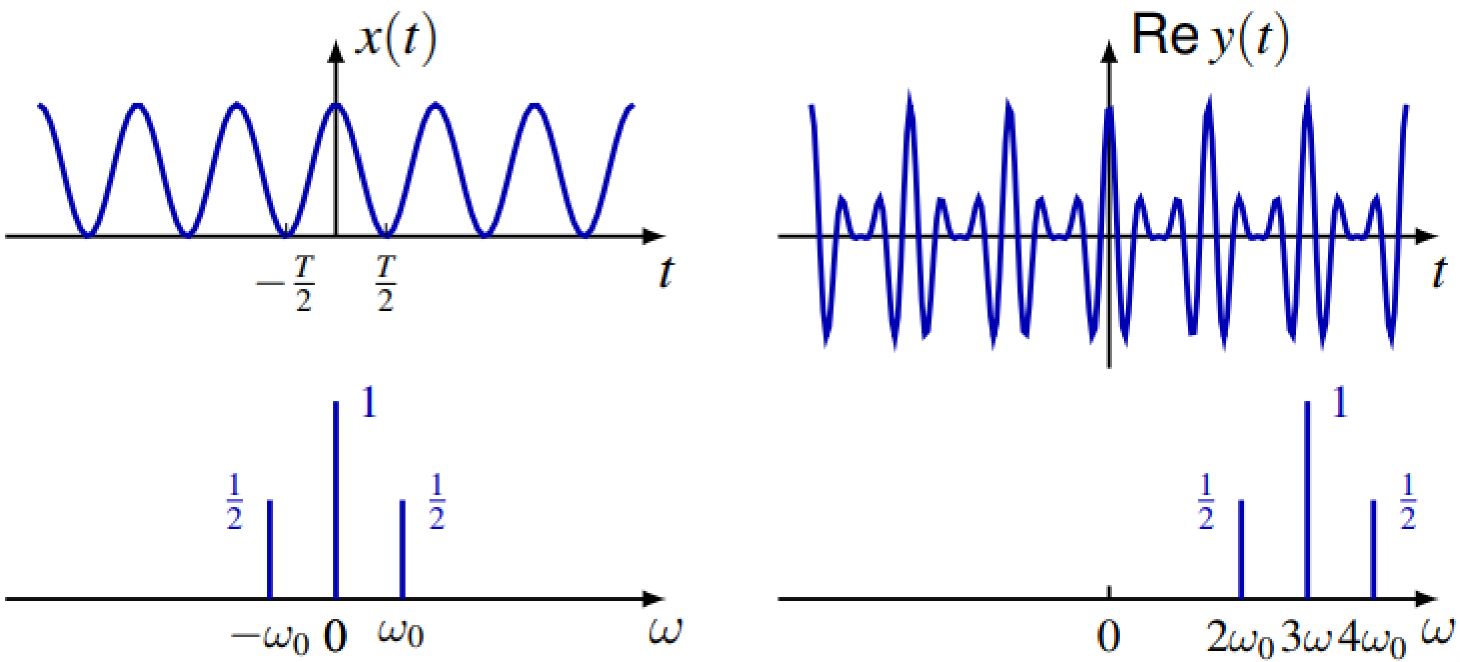
$$\begin{aligned} b_k &= \frac{1}{T}\int_T e^{jM\omega_0 t}x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_T x(t)e^{-j(k-M)\omega_0 t_0}dt \\ &= a_{k-M} \end{aligned}$$

- **Note:** Modulation by inharmonic exponential may change fundamental frequency or even result in aperiodic signal

Frequency Shifting

- **Example:** $x(t) = 1 + \cos(\omega_0 t) = 1 + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

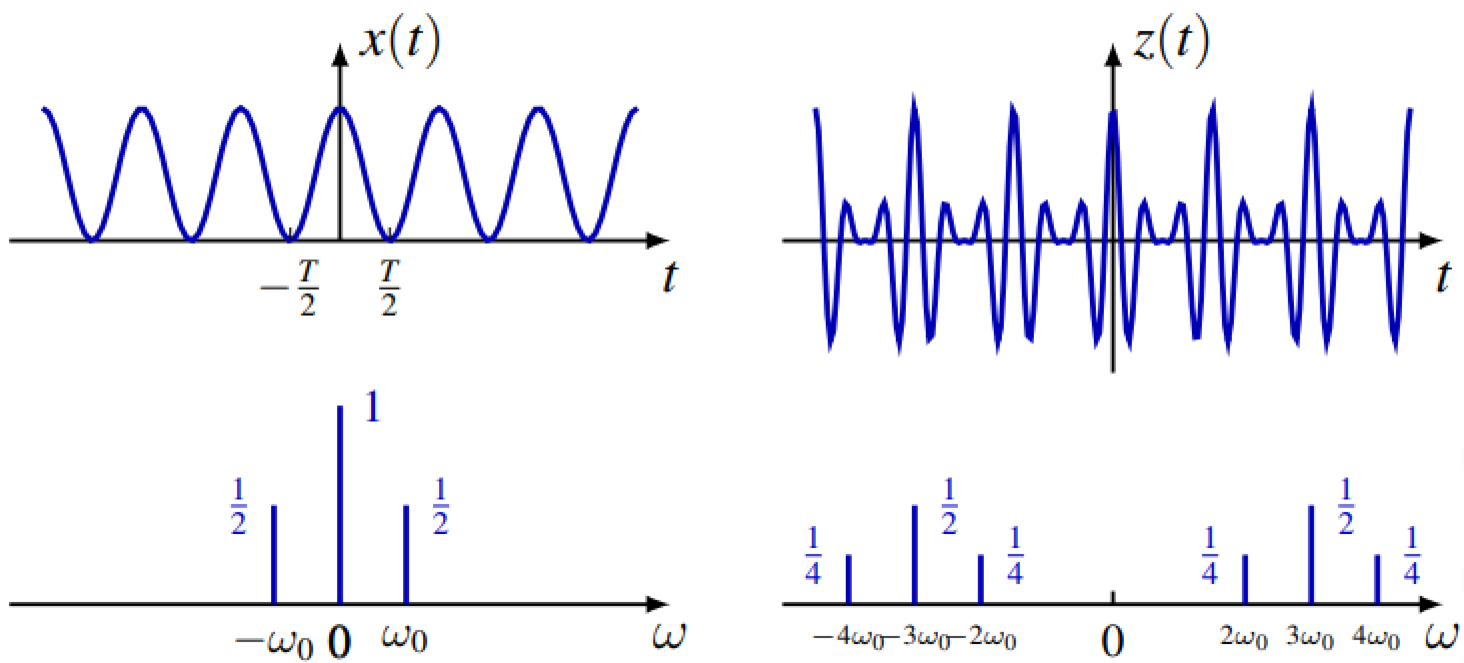
$$y(t) = e^{j3\omega_0 t}x(t) = e^{j3\omega_0 t} + \frac{1}{2}e^{j4\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t}$$



Frequency Shifting

- **Example:** $x(t) = 1 + \cos(\omega_0 t) = 1 + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$z(t) = \cos(3\omega_0 t)\, x(t) = \mathcal{R}e\{y(t)\}$$
$$= \frac{1}{2}e^{j3\omega_0 t} + \frac{1}{4}e^{j4\omega_0 t} + \frac{1}{4}e^{j2\omega_0 t} + \frac{1}{2}e^{-j3\omega_0 t} + \frac{1}{4}e^{-j4\omega_0 t} + \frac{1}{4}e^{-j2\omega_0 t}$$



Time Reversal

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, and

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\downarrow R \qquad\qquad \downarrow R$$

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_{-k}$$

Time reversal commutes with Fourier series

- **Proof:**

$$b_k = \frac{1}{T}\int_T x(-t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{T_R} x(\tau)e^{jk\omega_0\tau}d(-\tau) = \frac{1}{T}\int_T x(\tau)e^{-j(-k)\omega_0\tau}d\tau = a_{-k}$$

- **Corollary:** Fourier series preserves even/odd symmetry, i.e.,

$$x(t) \text{ even} \Leftrightarrow a_k \text{ even},\ x(t) \text{ odd} \Leftrightarrow a_k \text{ odd}$$

Time Scaling

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, and $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$x(\alpha t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \text{period } T/\alpha$$

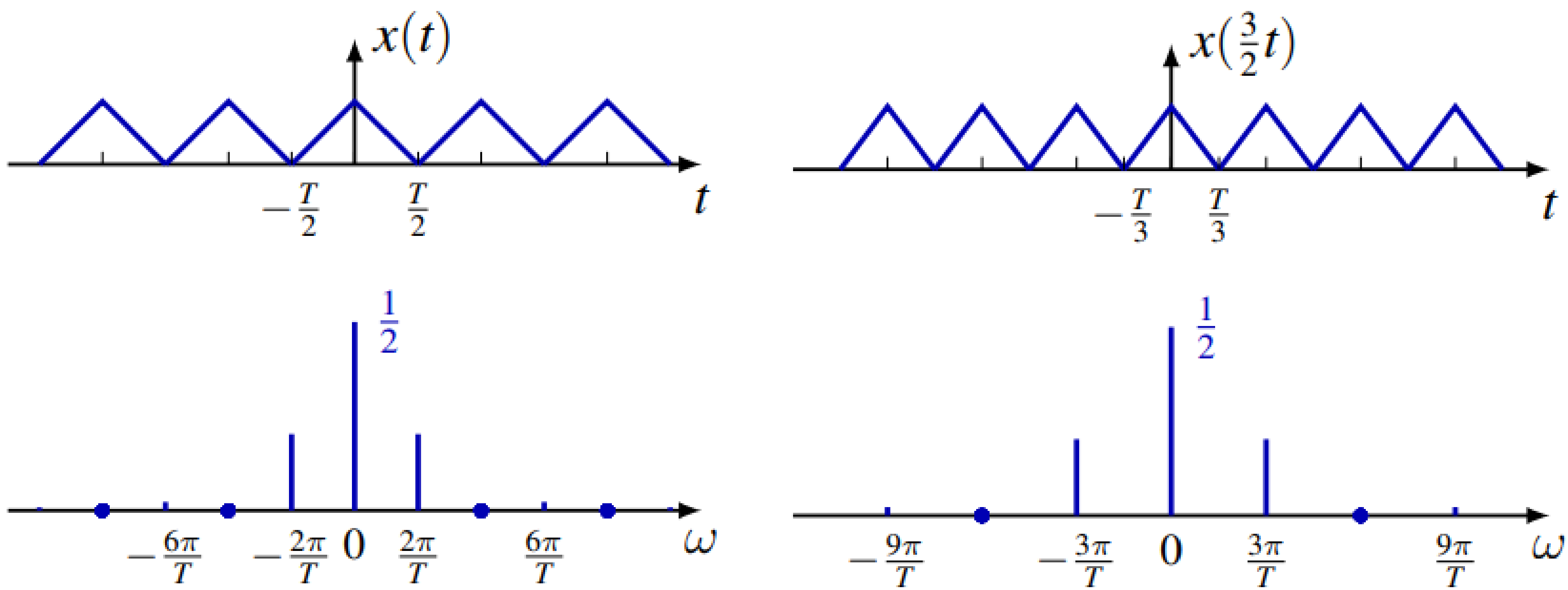
Time scaling preserves Fourier coefficients but changes fundamental frequency

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

- **Note:** Compression of a signal (i.e., $|\alpha| > 1$) in time domain results in the spectrum expansion, which is essentially a different Fourier series, and vice versa

Time Scaling

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{vs.} \qquad x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$



compression in time ⇔ expansion in frequency

expansion in time ⇔ compression in frequency

Differentiation

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, so does its derivative $x'(t)$, and if $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$x'(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = jk\omega_0 a_k$$

differentiation in time $\Leftrightarrow$ multiplication by $jk\omega_0$ in frequency

- **Proof:**

$$b_k = \frac{1}{T}\int_T x'(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{T}\left[x(t)e^{-jk\omega_0 t}\right]\Big|_0^T + jk\omega_0\left(\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t_0}dt\right) = jk\omega_0 a_k$$

- Alternatively, differentiate term by term

$$x(t) = \sum_k a_k e^{jk\omega_0 t} \implies x'(t) = \sum_k jk\omega_0 a_k e^{jk\omega_0 t}$$

Integration

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, and Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Integrating term by term

$$y(t) \triangleq \int_0^t x(s)\,ds = a_0 t + \sum_{k\neq 0} a_k \frac{e^{jk\omega_0 t} - 1}{jk\omega_0}$$

$$= a_0 t - \sum_{k\neq 0} \frac{a_k}{jk\omega_0} + \sum_{k\neq 0} \frac{a_k}{jk\omega_0} e^{jk\omega_0 t}$$

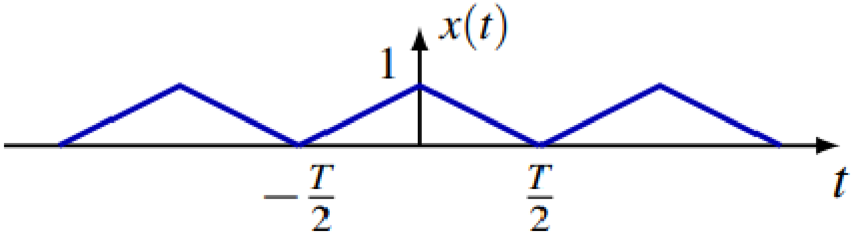
- $y(t)$ periodic i.f.f. $a_0 = 0$, i.e., $x(t)$ has no DC component
- if $a_0 = 0$, $y(t)$ has period T,

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \frac{a_k}{jk\omega_0}, \forall k \neq 0, \text{and} \int_{-\infty}^{t} x(s)\,ds = y(t) + \int_{-\infty}^{0} x(s)\,ds \rightarrow \text{constant}$$

Example: Triangle Wave

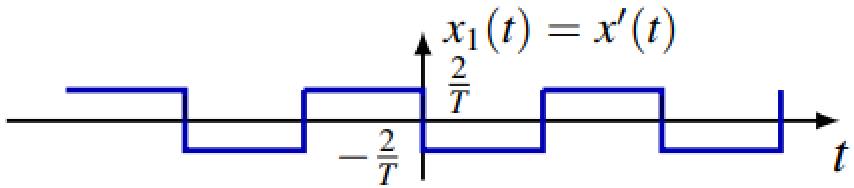
- $x(t)$: triangle wave

  $a_k = \frac{1}{jk\omega_0}\frac{4\sin(k\pi/2)}{k\pi \mathrm{T}}e^{jk\pi/2}, k \neq 0$



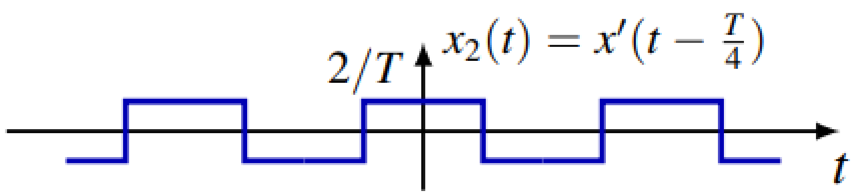
- $x_1(t) = x'(t)$

  $a_{1,k} = \frac{4\sin(k\pi/2)}{k\pi \mathrm{T}}e^{jk\pi/2} - \frac{2}{T}\delta[k]$



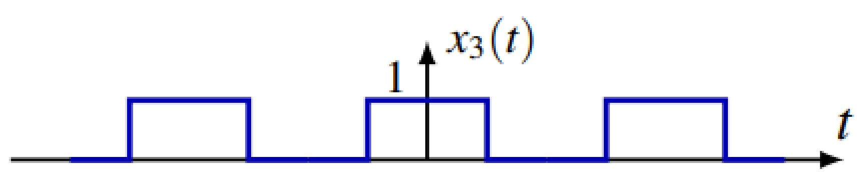
- $x_2(t) = x_1\left(t - \frac{T}{4}\right)$

  $a_{2,k} = \frac{4\sin(k\pi/2)}{k\pi \mathrm{T}} - \frac{2}{T}\delta[k]$



- $x_3(t) = \frac{T}{4}x_2(t) + \frac{1}{2}$

  $a_{3,k} = \frac{\sin(k\pi/2)}{k\pi}$



Multiplication

- If $x(t)$ and $y(t)$ have the **same** period T and $\omega_0 = 2\pi/T$, so does their product $x(t)y(t)$, and if $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} a_m b_{k-m}$$

multiplication in time ⇔ convolution in frequency

- **Proof:**

$$x(t)y(t) = \left(\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t}\right)\left(\sum_{l=-\infty}^{\infty} b_l e^{jl\omega_0 t}\right)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_m b_l e^{j(m+l)\omega_0 t} \qquad \textbf{(let } \boldsymbol{k = m + l}\textbf{)}$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} a_m b_{k-m}\right) e^{jk\omega_0 t}$$

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Periodic Convolution

- **Periodic Convolution** $x(t) * y(t)$ of $x(t)$ and $y(t)$ with the **same** period T and $\omega_0 = 2\pi/T$

$$x(t) * y(t) = \int_T x(\tau)y(t-\tau)d\tau$$

- **Properties**
  - Commutativity

$$x(t) * y(t) = y(t) * x(t)$$

  - Associativity

$$[x(t) * y(t)] * z(t) = x(t) * [y(t) * z(t)]$$

  - Bilinearity

$$\left(\sum_i A_i\, x_i(t)\right) * \left(\sum_j B_j\, y_j\,(t)\right) = \sum_{i,j} A_i\, B_j [x_i(t) * y_j(t)]$$

Periodic Convolution

- If $x(t)$ and $y(t)$ have the **same** period T and $\omega_0 = 2\pi/T$, and $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$

$$x(t) * y(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = T a_k b_k$$

convolution in time ⇔ multiplication in frequency

- **Proof:**

$$c_k = \frac{1}{T}\int_T [x(t) * y(t)] e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T \left[\int_T x(\tau) y(t-\tau) d\tau\right] e^{-jk\omega_0 t} dt$$

$$= \int_T x(\tau) \left[\frac{1}{T}\int_T y(t-\tau) e^{-jk\omega_0 t} dt\right] d\tau$$

$$= \int_T x(\tau) e^{-jk\omega_0 \tau} b_k\, d\tau = T a_k b_k$$

Conjugation and Symmetry

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, so does its complex conjugate $x^*(t) = \overline{x(t)}$, and if $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a^*_{-k}$$

- **Proof:**

$$b_k = \frac{1}{T}\int_T x^*(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T \overline{x(t)e^{jk\omega_0 t}}dt = \overline{a_{-k}} = a^*_{-k}$$

- **Corollary:** if $x(t)$ is real, then a_k is **conjugate symmetric**

$$a_{-k} = a_k^*$$

- **Corollary:** if $x(t)$ is real and even, then a_k is also real and even

$$a_k = a_{-k} = a_k^*$$

- **Corollary:** if $x(t)$ is real and odd, a_k is purely imaginary and odd

$$-a_k = a_{-k} = a_k^*$$

Even-Odd Decomposition

- If $x(t)$ is real with period T and $\omega_0 = 2\pi/T$, $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \overset{\mathcal{FS}}{\longleftrightarrow} b_k = \mathcal{Re}\{a_k\}$$
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \overset{\mathcal{FS}}{\longleftrightarrow} c_k = j \cdot \mathcal{Im}\{a_k\}$$

- **Proof:**
  - Since $x(t)$ is real, then a_k is conjugate symmetric

$$b_k = \frac{1}{2}[a_k + a_{-k}] = \frac{1}{2}[a_k + a_k^*] = \mathcal{Re}\{a_k\}$$
$$c_k = \frac{1}{2}[a_k - a_{-k}] = \frac{1}{2}[a_k - a_k^*] = j \cdot \mathcal{Im}\{a_k\}$$

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Parseval's Relation

- If $x(t)$ has period T and $\omega_0 = 2\pi/T$, $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- **Interpretation: Energy conservation**
  - $|a_k|^2$ is average power of k-th harmonic component
  - $\frac{1}{T}\int_T |x(t)|^2 dt$ is average power of $x(t)$ in one period, which equals the sum of average powers in all harmonic components
- **Hint for the Proof:**
  - Use the relation $|x(t)|^2 = x(t)x^*(t)$, and the properties of multiplication, conjugation and symmetry

DT Periodic Signals

- DT signal $x[n]$ is **periodic** with fundamental period N if

$$x[n+N] = x[n]$$

  - fundamental period N: smallest positive period
  - fundamental frequency: $\omega_0 = \begin{cases} \frac{2\pi}{N}, & N > 1 \\ 0, & N = 1 \end{cases}$

- Complex exponential $\phi_N^k[n] = e^{jk\frac{2\pi}{N}n} = e^{k\omega_0 n}$ is periodic with
  - period N and fundamental period $N/\gcd(N,k)$
  - fundamental frequency

$$\omega_k = \begin{cases} 0, & \text{if } N|k \\ \omega_0 \cdot \gcd(N,k), & \text{otherwise} \end{cases}$$

  - **always integer multiple of $\omega_0 = \frac{2\pi}{N}$, harmonically related**

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Finiteness of DT Fourier Basis

- **Fourier series** represents DT **signal with period *N*** in terms of **hamonically related** complex exponentials $\phi_N^k[n]$

$$x[n] = \sum_k a_k \, \phi_N^k[n] = \sum_k a_k \, e^{jk\frac{2\pi}{N}n}$$

- **Key difference with CT case**
  - $\phi_N^{k+rN}[n] = \phi_N^k[n] \Longrightarrow$ only ***N* distinct** $\phi_N^k[n]$, **finite** Fourier basis

$$\{\phi_N^k[n]: k \in \mathbb{Z}\} = \{\phi_N^k[n]: k \in \langle N \rangle\}$$

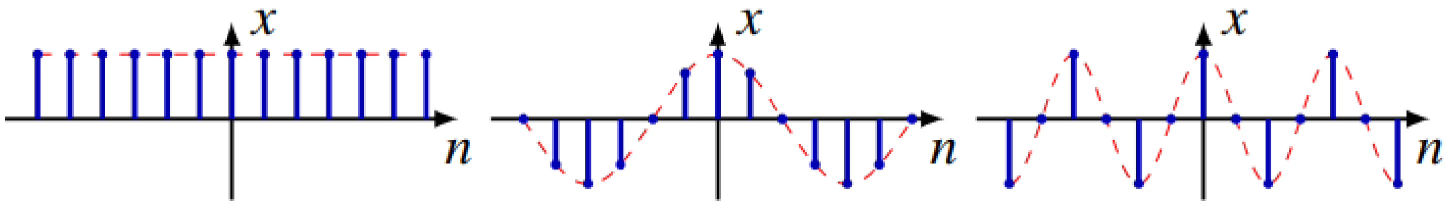
where $\langle N \rangle$ denotes the set of any N successive integers, e.g,.
$\langle N \rangle = \{0, 1, \ldots, \; N-1\}$, or $\langle N \rangle = \{2, 3, \ldots, \; N+1\}$

- **Proof:**

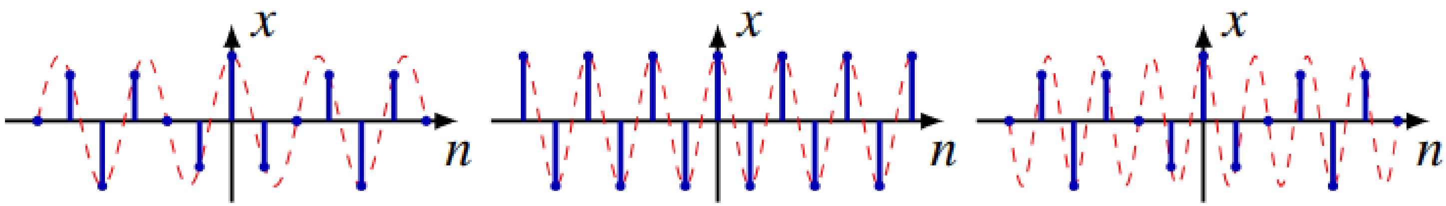
$$\phi_N^{k+rN}[n] = e^{j(k+rN)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} + e^{jr2\pi n} = \phi_N^k[n], \forall r \in \mathbb{Z}$$

Finiteness of DT Fourier Basis

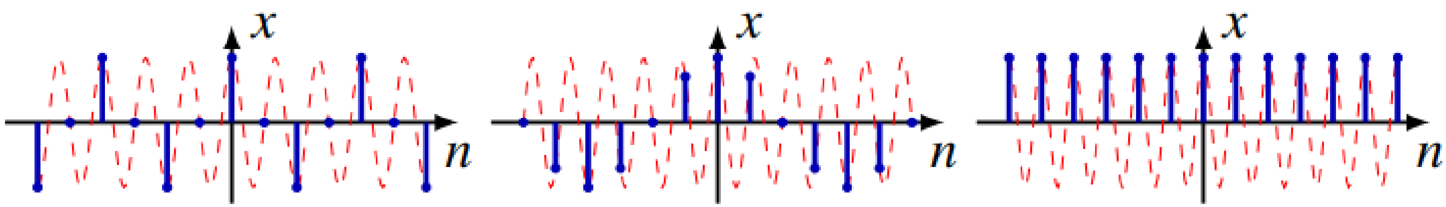
- $\phi_N^k[n]$ for $N = 4$ and $k = 0, 1, 2, \ldots, 8$



$\phi_N^0[n] = \cos(0 \cdot n) = 1$ $\phi_N^1[n] = \cos(\pi n/4)$ $\phi_N^2[n] = \cos(\pi n/2)$



$\phi_N^3[n] = \cos(3\pi n/4)$ $\phi_N^4[n] = \cos(\pi n)$ $\phi_N^5[n] = \cos(5\pi n/4)$



$\phi_N^6[n] = \cos(3\pi n/2)$ $\phi_N^7[n] = \cos(7\pi n/4)$ $\phi_N^8[n] = \cos(2\pi n) = 1$

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Orthonormality of Harmonics

- **DT Fourier series**

$$x[n] = \sum_{k\in\langle N\rangle} a_k\, \phi_N^k[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

  - Summation can take over **any N successive** integers
  - Coefficients a_k obtained using **orthonormality of harmonics**
- Define **inner product** between two signals with period N by

$$\langle x[n], y[n]\rangle = \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]\cdot\overline{y[n]}$$

  - Same as inner product in $\mathbb{C}^N$ up to factor N^{-1}
  - $\{\phi_N^k[n]: k\in\langle N\rangle\}$ is **orthonormal** system of functions

$$\langle \phi_N^k[n], \phi_N^m[n]\rangle = \delta[k-m]$$

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Orthonormality of Harmonics

- $\{\phi_N^k[n]: k \in \langle N \rangle\}$ is **orthonormal** system of functions

$$\langle \phi_N^k[n], \phi_N^m[n] \rangle = \delta[k-m]$$

- **Proof:**

$$\langle \phi_N^k[n], \phi_N^m[n] \rangle = \frac{1}{N} \sum_{n \in \langle N \rangle} e^{jk\frac{2\pi}{N}n} e^{-jm\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n}$$

  - if $k = m$, then

$$\langle \phi_N^k[n], \phi_N^m[n] \rangle = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1$$

  - if $k \neq m$, since $|k-m| \leq N-1$, $e^{j(k-m)\frac{2\pi}{N}n} \neq 1$, and by $\sum_{n=n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a}$, then

$$\langle \phi_N^k[n], \phi_N^m[n] \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n} = \frac{1}{\mathrm{N}} \frac{1 - e^{j(k-m)\frac{2\pi}{N}N}}{1 - e^{j(k-m)\frac{2\pi}{N}}} = 0$$

Determination of Fourier Coefficients

- Suppose $x[n]$ has period N and Fourier series representation

$$x[n] = \sum_{k\in\langle N\rangle} a_k\, \phi_N^k[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

- For $m \in \langle N\rangle$,

$$\langle x[n], \phi_N^m[n]\rangle = \left\langle \sum_{k\in\langle N\rangle} a_k\, \phi_N^k[n], \phi_N^m[n] \right\rangle = \sum_{k\in\langle N\rangle} a_k \langle \phi_N^k[n], \phi_N^m[n]\rangle$$

$$= \sum_{k\in\langle N\rangle} a_k\, \delta[k-m] = a_m$$

  - Can also think of a_k as a periodic signal with period N, since

$$a_{m+rN} \triangleq \langle x[n], \phi_N^{m+rN}[n]\rangle = \langle x[n], \phi_N^m[n]\rangle = a_m, \forall r \in \mathbb{Z}$$

  - But **only N successive values** used in Fourier series!

DT Fourier Series

- **Synthesis equation**

$$x[n] = \sum_{k\in\langle N\rangle} a_k\, \phi_N^k[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\omega_0 n} = \sum_{k\in\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

- **Analysis equation**

$$a_k = \langle x[n], \phi_N^k[n]\rangle = \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$$

- **Note: no convergence issue since all sums are finite!**

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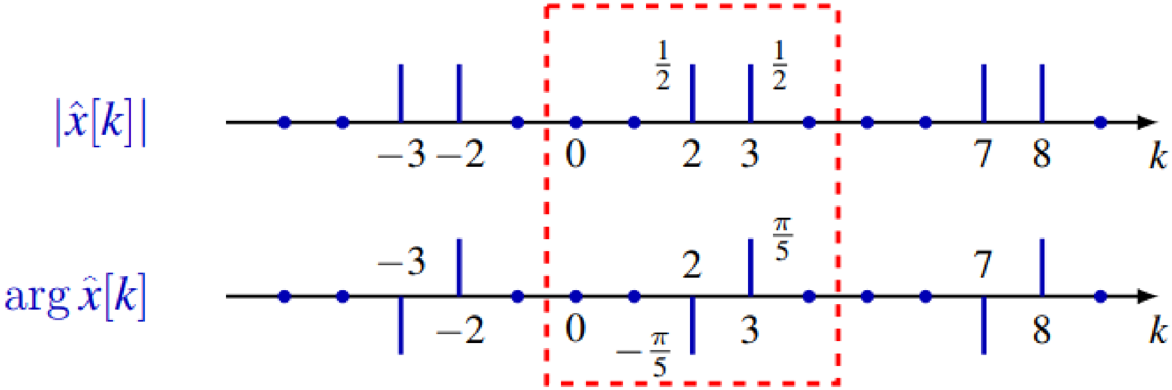
Example: Sinusoidal Function

$$x[n] = \cos\left(\frac{6\pi}{5}n + \frac{\pi}{5}\right), \text{ with period } N = 5$$

$$\Rightarrow \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5}, x[n] = \frac{1}{2}e^{j\left(\frac{6\pi}{5}n+\frac{\pi}{5}\right)} + \frac{1}{2}e^{-j\left(\frac{6\pi}{5}n+\frac{\pi}{5}\right)} = \frac{e^{j\frac{\pi}{5}}}{2}e^{j3\frac{2\pi}{5}n} + \frac{e^{-j\frac{\pi}{5}}}{2}e^{-j3\frac{2\pi}{5}n}$$

- **Fourier coefficients** a_k repeat with period $N = 5$
  - $a_3 = \frac{1}{2}e^{j\frac{\pi}{5}}, a_2 = a_{2-5} = a_{-3} = \frac{1}{2}e^{-j\frac{\pi}{5}}, a_0 = a_1 = a_4 = 0$



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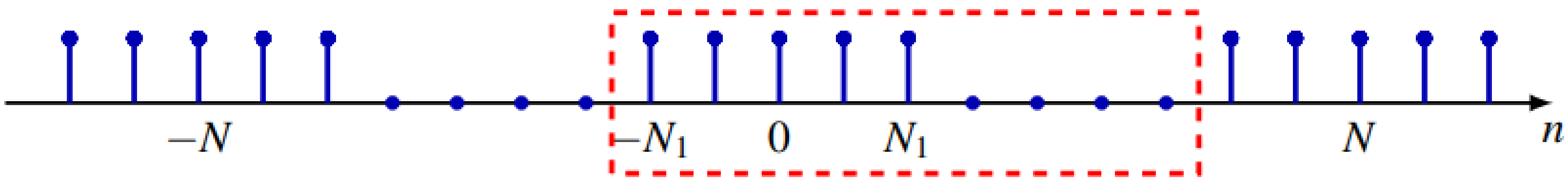
Example: Periodic Square Wave

- Periodic square wave with period N, in one period

$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & N_1 < n < N - N_1 \end{cases}$$



- **Fourier coefficients**

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n}$$

- Case 1: if k is integer multiple of N,

$$a_k = \frac{2N_1 + 1}{N}$$

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Example: Periodic Square Wave

- Case 2: if k is not integer multiple of N, then $e^{-jk\frac{2\pi}{N}} \neq 1$, using

$$\sum_{n=m}^{M} a^n = \frac{a^m - a^{M+1}}{1-a}$$

we obtain

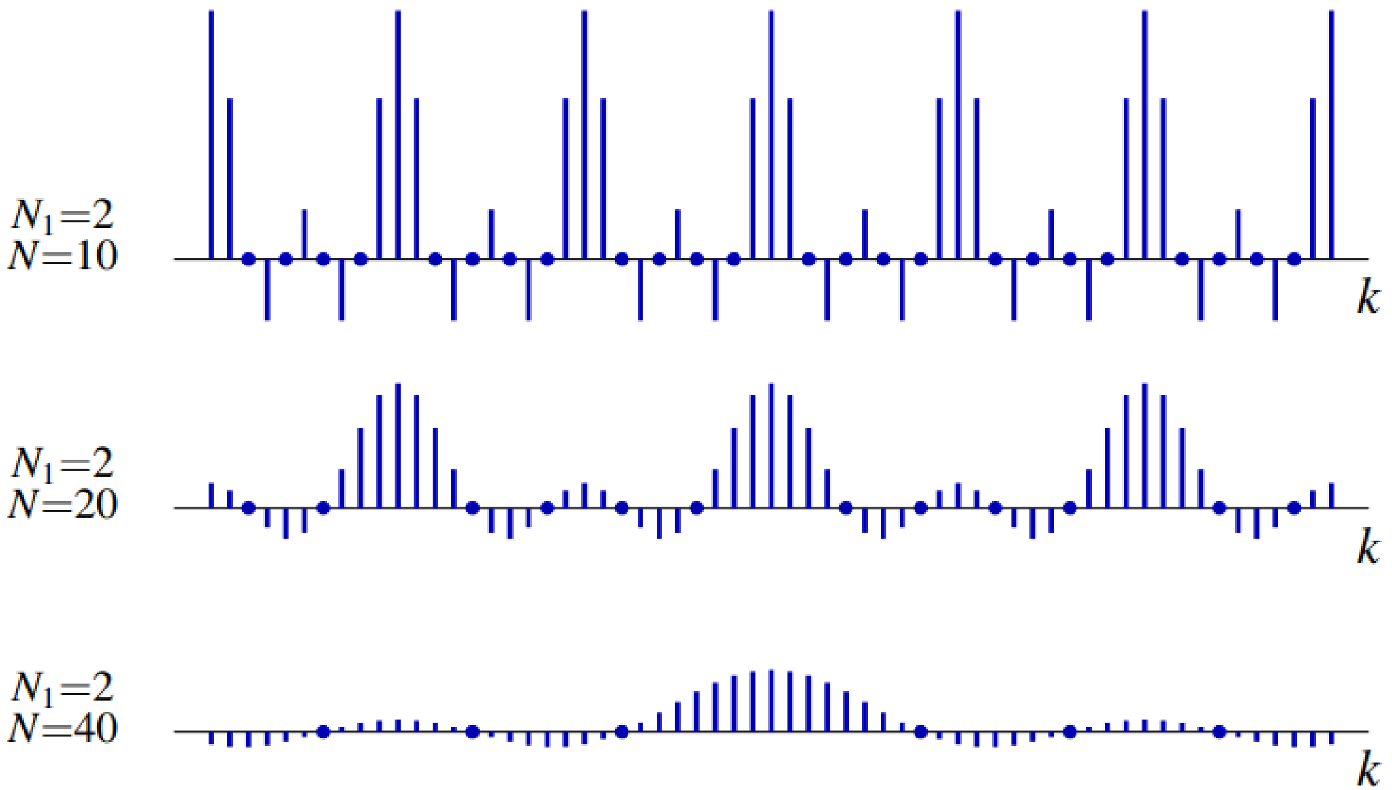
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\frac{e^{jk\frac{2\pi}{N}N_1} - e^{-jk\frac{2\pi}{N}(N_1+1)}}{1-e^{-jk\frac{2\pi}{N}}}$$

$$= \frac{1}{N}\frac{e^{jk\frac{2\pi}{N}(N_1+1/2)} - e^{-jk\frac{2\pi}{N}(N_1+1/2)}}{e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}}$$

$$= \frac{1}{N}\frac{\sin\left[k\frac{2\pi}{N}(N_1+1/2)\right]}{\sin\left(k\frac{\pi}{N}\right)}$$

Example: Periodic Square Wave

$$a_k = \frac{1}{N} \frac{\sin\left[k\frac{2\pi}{N}(N_1+1/2)\right]}{\sin\left(k\frac{\pi}{N}\right)}$$



DT Fourier Series

- **DT Fourier series** of $x[n]$ with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\omega_0 n} = \sum_{k\in\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

- Correspondence between two periodic functions with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

- Two equivalent representations of same signal
  - **time domain**: $x[n]$
  - **frequency domain**: a_k

Properties of DT Fourier Series

- **Linearity**
  - If $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k,\ y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$ have same period N

$$Ax[n] + By[n] \overset{\mathcal{FS}}{\longleftrightarrow} Aa_k + Bb_k$$

- **Time and frequency shifting**
  - If $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ has period N and $\omega_0 = 2\pi/N$

$$x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 n_0} a_k$$

and

$$e^{jm\omega_0} x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_{k-m}$$

Properties of DT Fourier Series

If $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ has period N and $\omega_0 = 2\pi/N$

- **Time reversal**

$$x[-n] \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$$

- **Conjugation**

$$x^*[n] \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$$

- **Symmetry**
  - $x[n]$ even $\Leftrightarrow a_k$ even, $x[n]$ odd $\Leftrightarrow a_k$ odd
  - $x[n]$ real $\Leftrightarrow a_{-k} = a_k^*$
  - $x[n]$ real and even $\Leftrightarrow a_k$ real and even
  - $x[n]$ real and odd $\Leftrightarrow a_k$ purely imaginary and odd

Time Scaling

- Define $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is integer multiple of } m \\ 0, & \text{otherwise} \end{cases}$
- If $x[n]$ has period N, $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$, then $x_{(m)}[n]$ has period mN

$$x_{(m)}[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k = \frac{1}{m} a_k$$

- **Proof:**

$$b_k = \frac{1}{mN} \sum_{n \in \langle mN \rangle} x_{(m)}[n] e^{-jk\frac{2\pi}{mN}n} = \frac{1}{mN} \sum_{l \in \langle N \rangle} x[l] e^{-jk\frac{2\pi}{N}l} = \frac{1}{m} a_k$$

- **Note:** $x_{(m)}[n]$ and b_k have period mN, so

$$x_{(m)}[n] = \sum_{k \in \langle mN \rangle} \frac{1}{m} a_k e^{jk\frac{2\pi}{mN}n}$$

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First Difference and Running Sum

If $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ has period N and $\omega_0 = 2\pi/N$

- **First difference** (analog of derivative for CT signals)
  - $\Delta x[n] = x[n] - x[n-1]$ has period N

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} (1 - e^{-jk\frac{2\pi}{N}})a_k$$

- **Running sum** (analog of integration for CT signals)

$$y[n] = \sum_{m=n_0}^{n} x[m]$$

  - $y[n]$ is periodic i.f.f. $a_0 = 0$, i.e., $x[n]$ has no DC component
  - if $a_0 = 0$, $y[n]$ also has period N

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{1-e^{-jk\frac{2\pi}{N}}} a_k, \ \text{ for } k \neq 0$$

Multiplication

- If $x[n]$ and $y[n]$ have the **same** period N and $\omega_0 = 2\pi/N$, so does their product $x[n]y[n]$, and if $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ and $y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{m\in\langle N\rangle} a_m\, b_{k-m}$$

- **Note:** Frequency domain: **periodic** convolution in DT case vs. **aperiodic** convolution in CT case
- **Proof:**

$$x[n]y[n] = \left(\sum_{m\in\langle N\rangle} a_m e^{jm\omega_0 n}\right)\left(\sum_{l\in\langle N\rangle} b_l e^{jl\omega_0 n}\right)$$

$$= \sum_{m\in\langle N\rangle}\sum_{l\in\langle N\rangle} a_m b_l e^{j(m+l)\omega_0 n} \quad \left(\begin{array}{l}\text{let } k = m + l \\ \text{use arithmetic mode } N\end{array}\right)$$

$$= \sum_{k\in\langle N\rangle}\left(\sum_{m\in\langle N\rangle} a_m b_{k-m}\right) e^{jk\omega_0 n}$$

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Periodic Convolution

- **Periodic Convolution** $x[n] * y[n]$ of $x[n]$ and $y[n]$ with the **same** period N and $\omega_0 = 2\pi/N$

$$x[n] * y[n] = \sum_{m\in\langle N\rangle} x[m]y[n-m]$$

- **Properties**
  - Commutativity

$$x[n] * y[n] = y[n] * x[n]$$

  - Associativity

$$(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$$

  - Bilinearity

$$\left(\sum_i A_i\, x_i[n]\right) * \left(\sum_j B_j\, y_j\,[n]\right) = \sum_{i,j} A_i\, B_j (x_i[n] * y_j[n])$$

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Periodic Convolution

- If $x[n]$ and $y[n]$ have the **same** period N and $\omega_0 = 2\pi/N$, and $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ and $y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$

$$x[n] * y[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k = N a_k b_k$$

convolution in time ⇔ multiplication in frequency

- **Proof:**

$$c_k = \frac{1}{N}\sum_{n\in\langle N\rangle}(x[n]*y[n])e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n\in\langle N\rangle}\left(\sum\nolimits_{m\in\langle N\rangle} x[m]y[n-m]\right)e^{-jk\frac{2\pi}{N}n}$$

$$= \sum_{m\in\langle N\rangle} x[m]\left(\frac{1}{N}\sum\nolimits_{n\in\langle N\rangle} y[n-m]\,e^{-jk\frac{2\pi}{N}n}\right)$$

$$= N\frac{1}{N}\sum_{m\in\langle N\rangle} x[m]e^{-jk\frac{2\pi}{N}m} b_k = N a_k b_k$$

Parseval's Relation

- If $x[n]$ has period N and $\omega_0 = 2\pi/N$, $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$

$$\frac{1}{N}\sum_{n\in\langle N\rangle} |x[n]|^2 = \sum_{k\in\langle N\rangle} |a_k|^2$$

- **Interpretation: Energy conservation**
  - $|a_k|^2$ is average power of k-th harmonic component
  - $\frac{1}{N}\sum_{n\in\langle N\rangle}|x[n]|^2$ is average power of $x[n]$ in one period, which equals the sum of average powers in all N distinct harmonic components
- **Hint for the Proof:**
  - Use the relation $|x[n]|^2 = x[n]x^*[n]$, and the properties of multiplication, conjugation and symmetry

Eigenvalues and Eigenfunctions of LTI Systems

- **CT Exponential $x(t) = e^{st}$**

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)}\, d\tau = e^{st} \int h(\tau)\, e^{-s\tau} d\tau \triangleq H(s) e^{st}$$

  - e^{st} is **eigenfunction** of LTI system with associated **eigenvalue**

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau \qquad \textbf{(system function)}$$

  - Restricted to $s = j\omega \implies$ **Frequency response $H(j\omega)$**

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega t} dt$$

- **DT Exponential $x[n] = z^n$**

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k} \triangleq H(z) z^n$$

  - z^n is **eigenfunction** of LTI system with associated **eigenvalue**

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k} \qquad \textbf{(system function)}$$

  - Restricted to $z = e^{j\omega} \implies$ **Frequency response $H(e^{j\omega})$**

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$$

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Frequency Responses

- **Fourier Series** can construct **any CT periodic signal** and essentially **all practically important periodic CT signals**
- CT LTI systems

$$x(t)=\sum_k a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\sum_k H(jk\omega_0)\, a_k e^{jk\omega_0 t}$$

- DT LTI systems

$$x[n]=\sum_{k\in\langle N\rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\sum_{k\in\langle N\rangle} H(e^{jk\omega_0}) a_k e^{jk\omega_0 n}$$

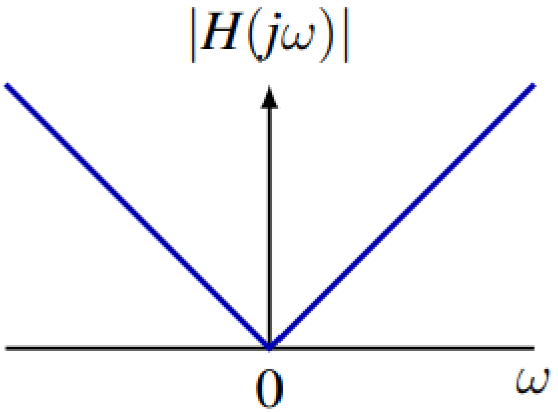
- Output periodic with same period of input, Fourier coefficients

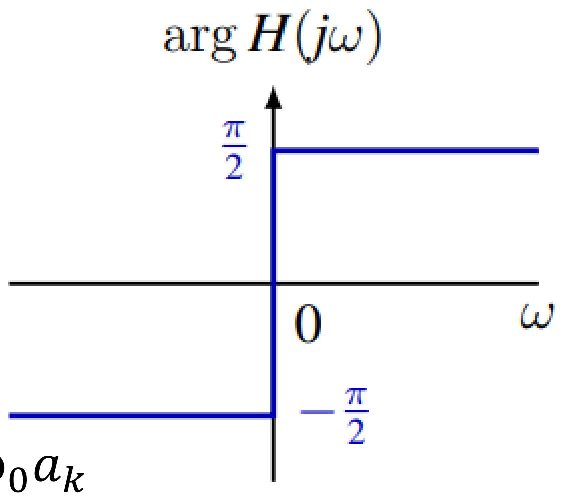
$$y(t)\overset{\mathcal{FS}}{\longleftrightarrow} H(jk\omega_0)a_k, \qquad y[n]\overset{\mathcal{FS}}{\longleftrightarrow} H\left(e^{jk\omega_0}\right)a_k$$

- **Filtering** changes relative amplitudes of frequency components or eliminates some frequency components entirely

CT Filtering

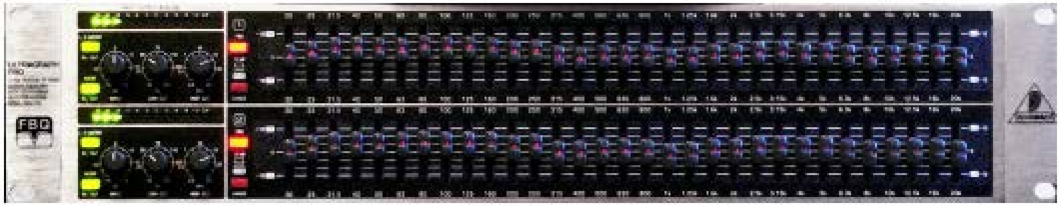
- **Example:** Differentiator $y(t) = x'(t)$
  - Impulse response
    - $h(t) = \delta'(t)$
  - Frequency response
    - $H(j\omega) = j\omega$
  - For periodic input
    - $x(t) = \sum_k a_k e^{jk\omega_0 t}$
  - Output
    - $y(t) = \sum_k jk\omega_0 a_k e^{jk\omega_0 t}$
  - Fourier coefficients rerlated by $b_k = jk\omega_0 a_k$





CT Filtering

- **LTI systems as filters**
  - cannot create new frequency components
  - can only scale magnitudes or shift phases of existing components
- **Example of nonlinear filter:** Clipping $y(t) = \max\{x(t), c\}$
- Focus on LTI filters in this course
- **Frequency-shaping filters** change shape of spectrum
  - e.g., equalizer



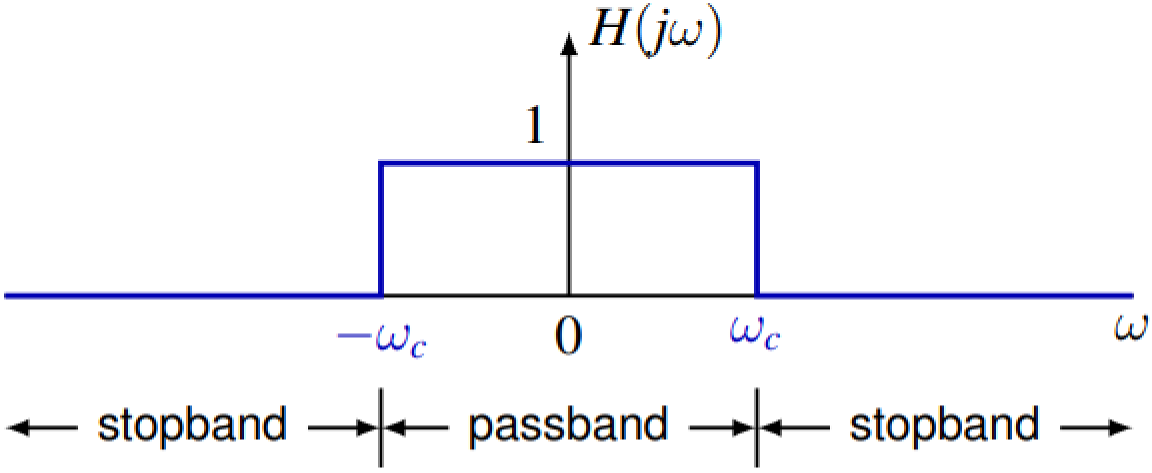
- **Frequency-selective filters** pass some frequencies essentially undistorted and significantly attenuate or eliminate others

Ideal Frequency-selective Filters

- **Ideal lowpass filter**

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

- $\boldsymbol{\omega_c}$: **cutoff frequency**

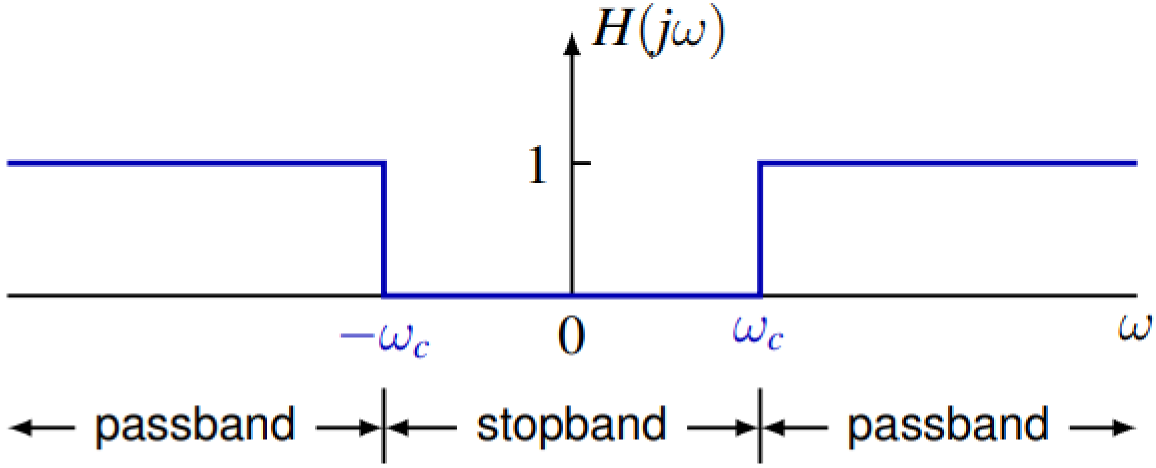


Ideal Frequency-selective Filters

- **Ideal highpass filter**

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

- $\boldsymbol{\omega_c}$**: cutoff frequency**

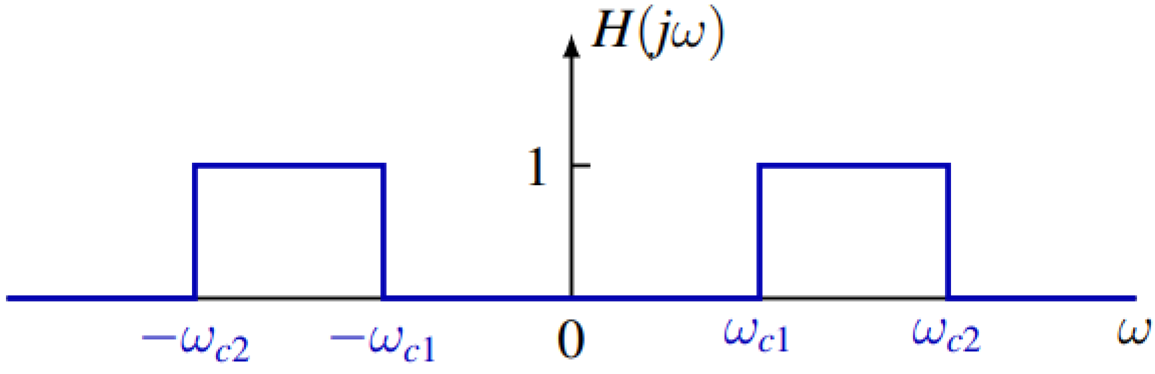


Ideal Frequency-selective Filters

- **Ideal bandpass filter**

$$H(j\omega) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

- ω_{c1}: **lower cutoff frequency**
- ω_{c2}: **upper cutoff frequency**

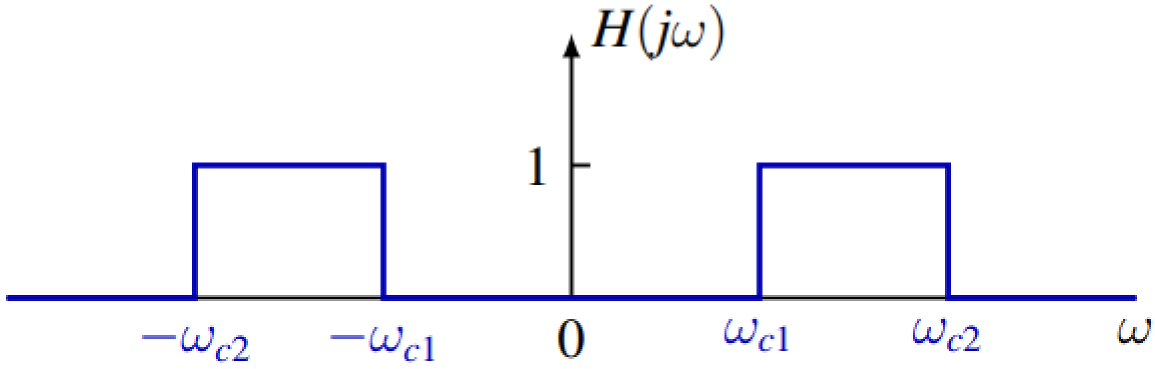


Ideal Frequency-selective Filters

- **Ideal bandpass filter**

$$H(j\omega) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

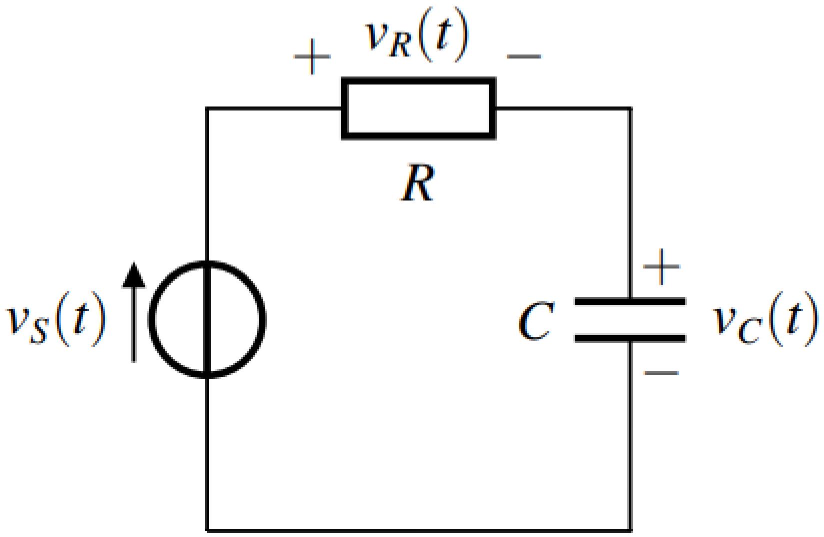
- $\boldsymbol{\omega_{c1}}$: **lower cutoff frequency**
- $\boldsymbol{\omega_{c2}}$: **upper cutoff frequency**



Simple RC Lowpass Filter

- **Differential Equation**

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$



- For input

$$v_s(t) = e^{j\omega t}$$

- Output

$$v_c(t) = H(j\omega)e^{j\omega t}$$

- **Frequency response:**

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

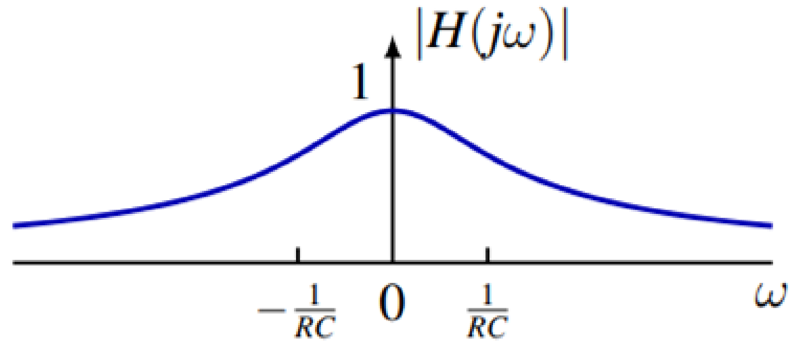
Simple RC Lowpass Filter

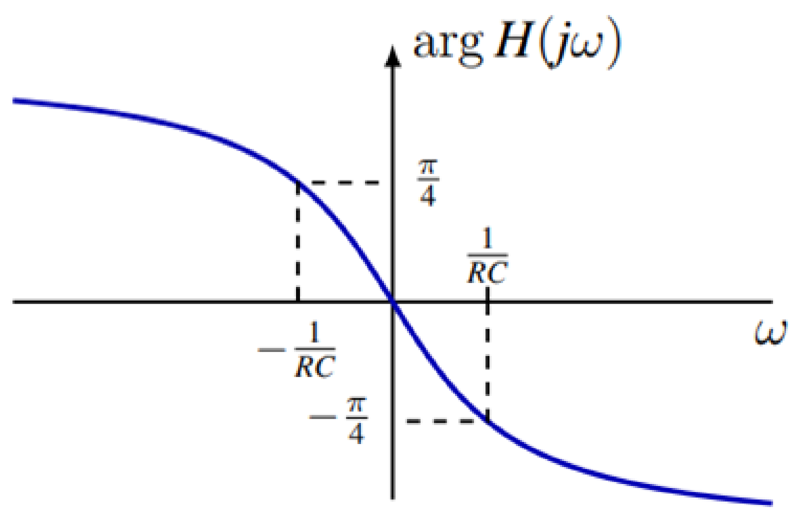
- **Frequency response**

$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

$$\arg H(j\omega) = -\arctan(RC\omega)$$





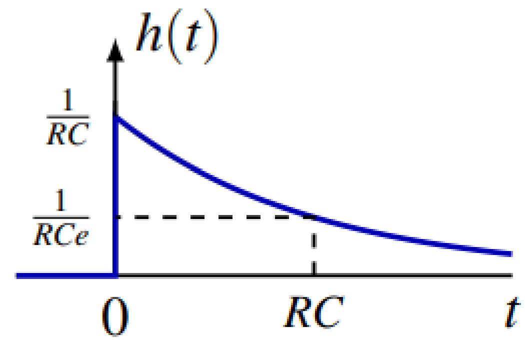
- **Nonideal lowpass filter**
  - passes lower frequencies
  - attenuates higher frequencies

- Larger RC $\Longrightarrow$ passes smaller range of lower frequencies

Simple RC Lowpass Filter

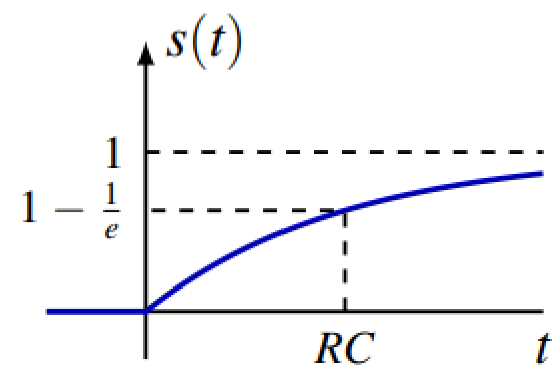
- **Impulse response**

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



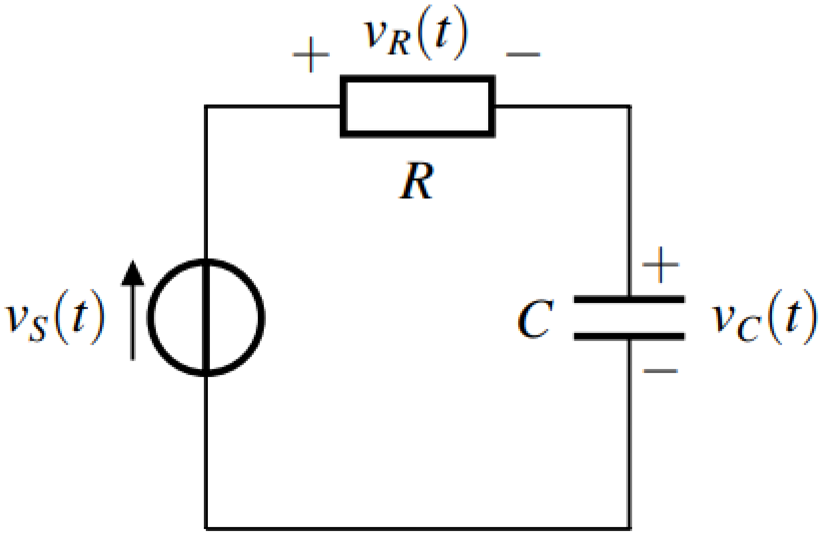
- **Step response**

$$s(t) = h(t) * u(t) = (1 - e^{-t/RC})u(t)$$



- **Time constant** $\tau = RC$
  - **Tradeoff:**
    - larger τ , passes fewer higher frequencies, more sluggish response
    - smaller τ , passes more higher frequencies, faster response

Simple RC Highpass Filter



- **Differential Equation**

$$RC\frac{dv_R(t)}{dt} + v_R(t) = RC\frac{dv_s(t)}{dt}$$

- For input

$$v_s(t) = e^{j\omega t}$$

- Output

$$v_R(t) = H(j\omega)e^{j\omega t}$$

- **Frequency response:**

$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

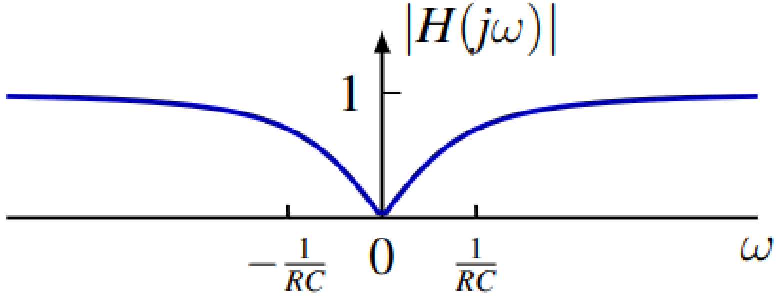
Simple RC Highpass Filter

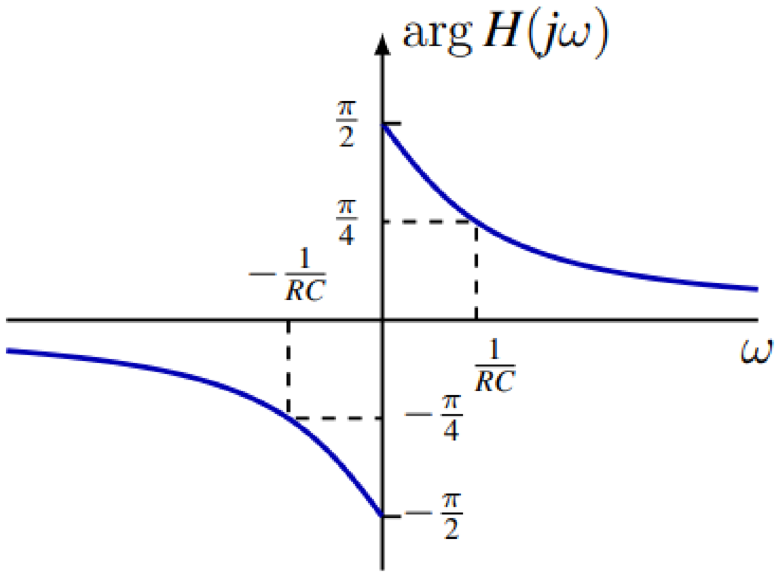
- **Frequency response**

$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$|H(j\omega)| = \frac{|\omega| RC}{\sqrt{1 + (RC\omega)^2}}$$

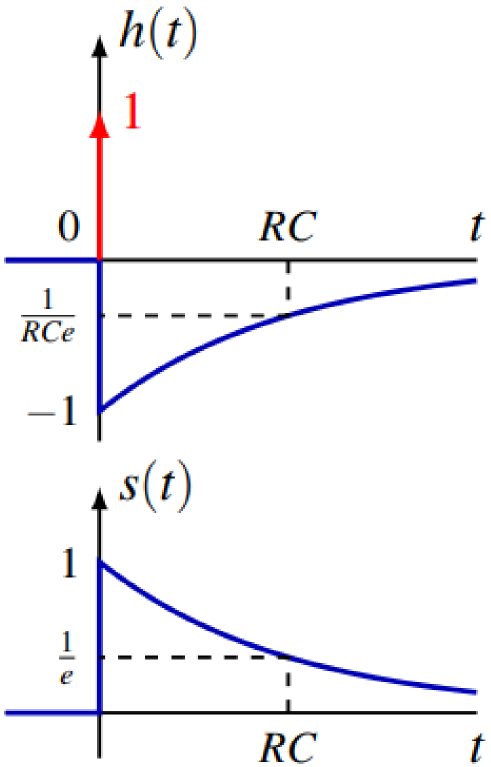
$$\arg H(j\omega) = \arctan\left(\frac{1}{RC\omega}\right)$$





- **Nonideal highpass filter**
  - passes higher frequencies
  - attenuates lower frequencies
- Larger RC $\Longrightarrow$ passes larger range of lower frequencies

Simple RC Highpass Filter



- **Step response**

  $$s(t) = e^{-t/RC}u(t)$$

- **Impulse response**

  $$h(t) = s'(t) = \delta(t) - \frac{1}{RC}e^{-t/RC}u(t)$$

- **Time constant** $\tau = RC$
  - **Tradeoff:**
    - larger τ , passes more lower frequencies, more sluggish response
    - smaller τ , passes fewer lower frequencies, faster respons

DT Filtering

- **Frequency-shaping** vs. **frequency-selective filters** as in CT case

- **LTI systems as filters**
  - cannot create new frequency components
  - can only scale magnitudes or shift phases of existing components

- **Example of nonlinear filter**
  - Max filter: $y[n] = \max\limits_{-n_1 \le k \le n_2} x[n+k]$
  - Median filter: $y[n] = \text{median}\{x[n-n_1], \ldots, \ x[n+n_2]\}$

- Recall for DT signals, suffices to consider frequencies on an interval of length 2π, e.g., $[0, 2\pi)$ or $(-\pi, \pi]$

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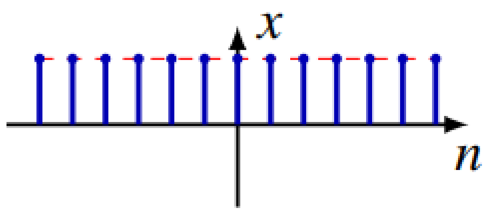
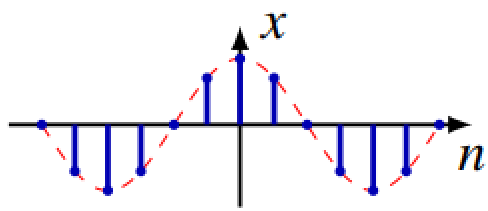
High vs. Low Frequencies for DT Signals

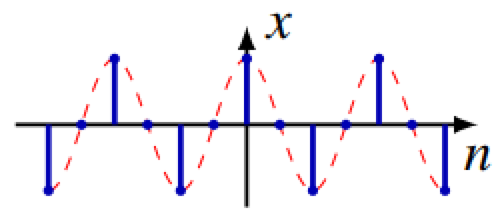
- High frequencies around $(2k+1)\pi$, low frequencies around $2k\pi$



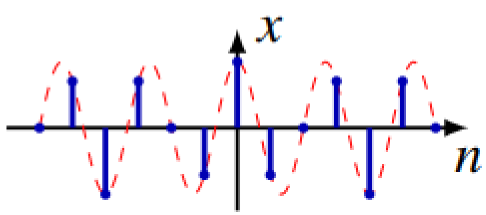
$\phi_N^0[n] = \cos(0 \cdot n) = 1$



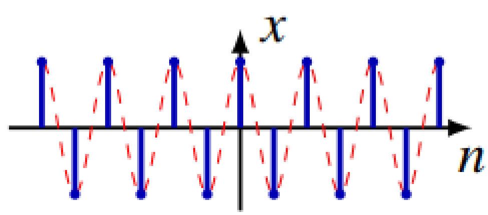
$\phi_N^1[n] = \cos(\pi n/4)$



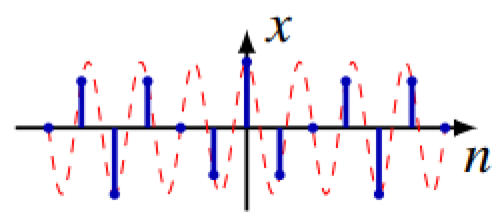
$\phi_N^2[n] = \cos(\pi n/2)$



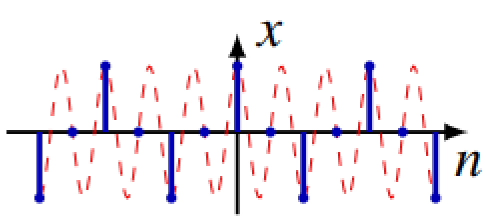
$\phi_N^3[n] = \cos(3\pi n/4)$



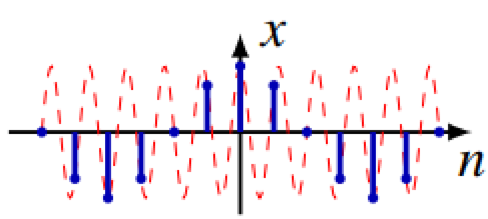
$\phi_N^4[n] = \cos(\pi n)$



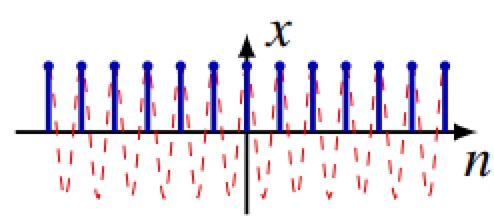
$\phi_N^5[n] = \cos(5\pi n/4)$



$\phi_N^6[n] = \cos(3\pi n/2)$



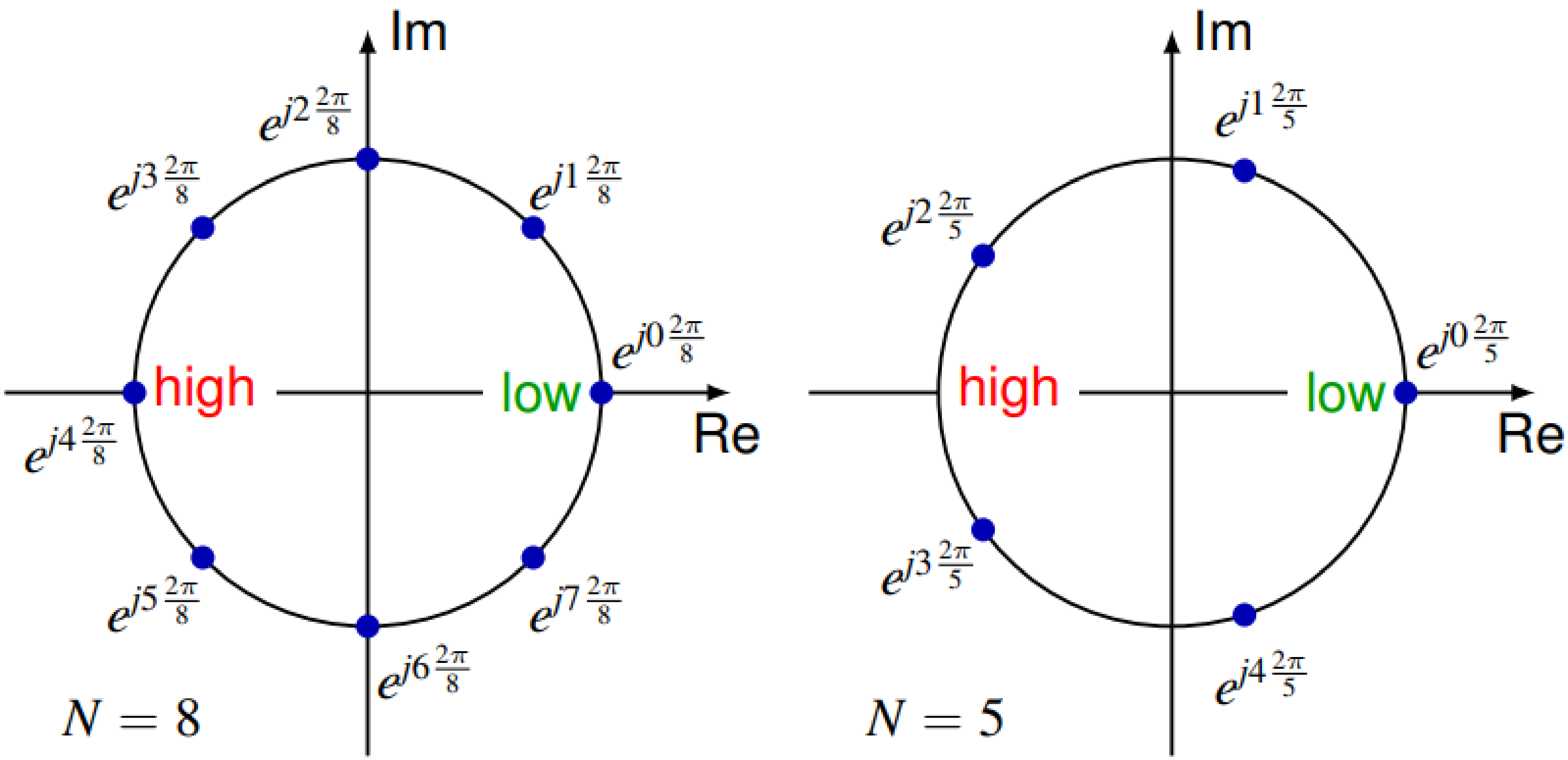
$\phi_N^7[n] = \cos(7\pi n/4)$



$\phi_N^8[n] = \cos(2\pi n) = 1$

DT Frequencies

- Discrete frequencies of periodic signals with period N
  - evenly spaced points on unit circle
  - low frequencies close to 1; high frequencies close to -1

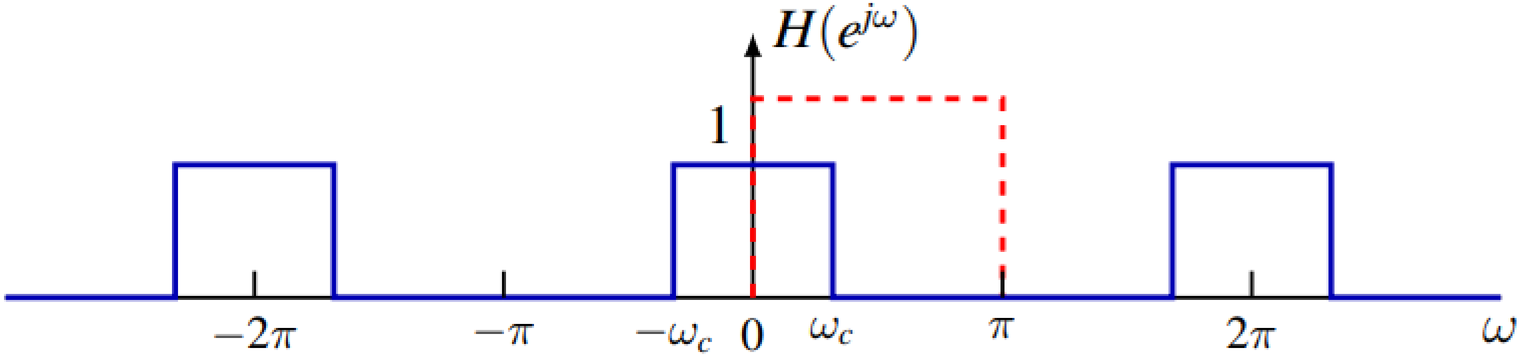


Ideal Frequency-selective Filters

- **Ideal lowpass filter**

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- $\boldsymbol{\omega_c}$**: cutoff frequency**

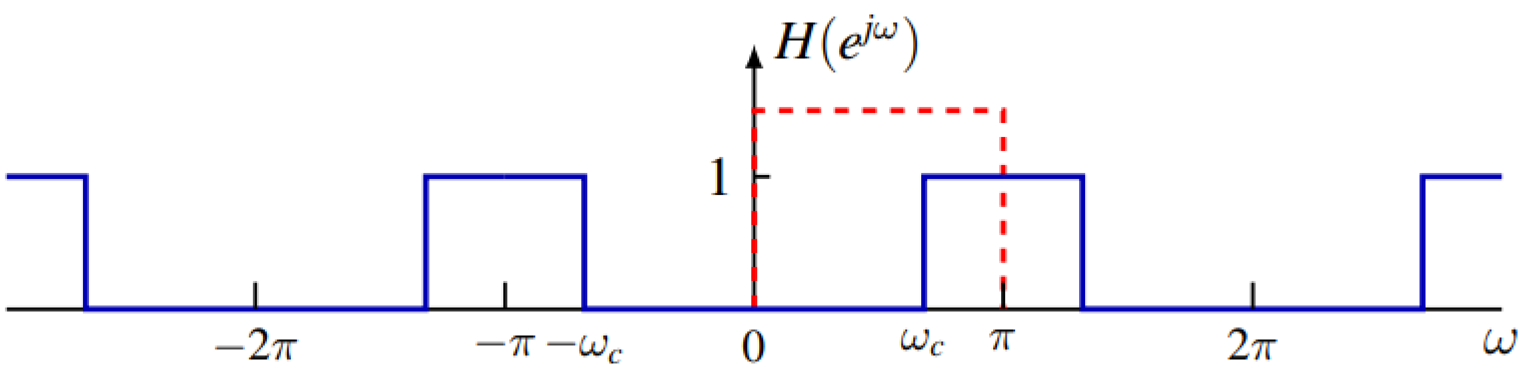


Ideal Frequency-selective Filters

- **Ideal highpass filter**

$$H(e^{j\omega}) = \begin{cases} 1, & \omega_c \le |\omega| \le \pi \\ 0, & |\omega| < \omega_c \end{cases}$$

- $\boldsymbol{\omega_c}$: **cutoff frequency**

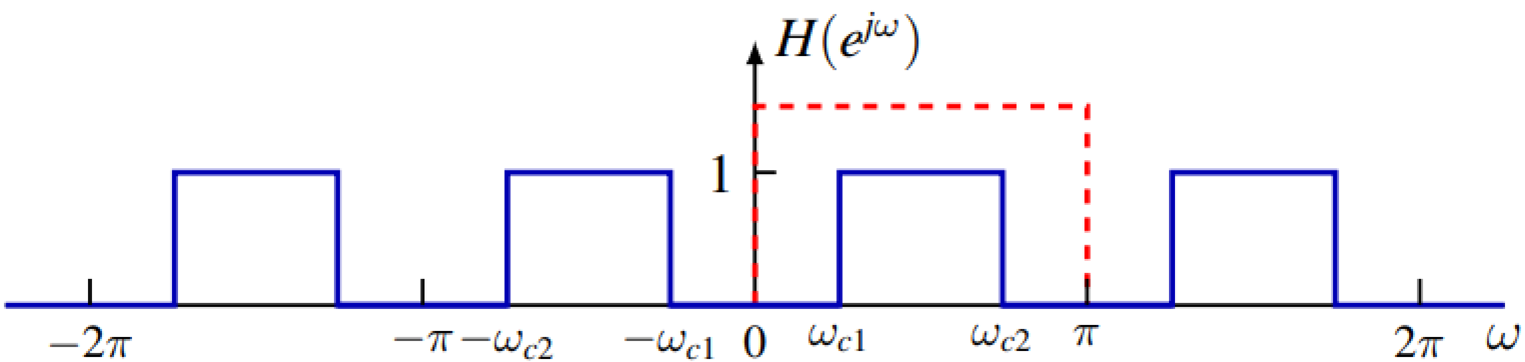


Ideal Frequency-selective Filters

- **Ideal bandpass filter**

$$H(e^{j\omega}) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & |\omega| < \omega_{c1} \text{ or } \omega_{c2} < |\omega| < \pi \end{cases}$$

- $\boldsymbol{\omega_{c1}}$: **lower cutoff frequency**
- $\boldsymbol{\omega_{c2}}$: **upper cutoff frequency**



First-order Recursive DT Filters

$$y[n] - ay[n-1] = x[n]$$

For input $x[n] = e^{j\omega n}$, output $y[n] = H(e^{j\omega})e^{j\omega n}$

- **Frequency response** (well defined if $|a| < 1$)

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}, |a| < 1$$

- For $a = |a|e^{j\phi}$

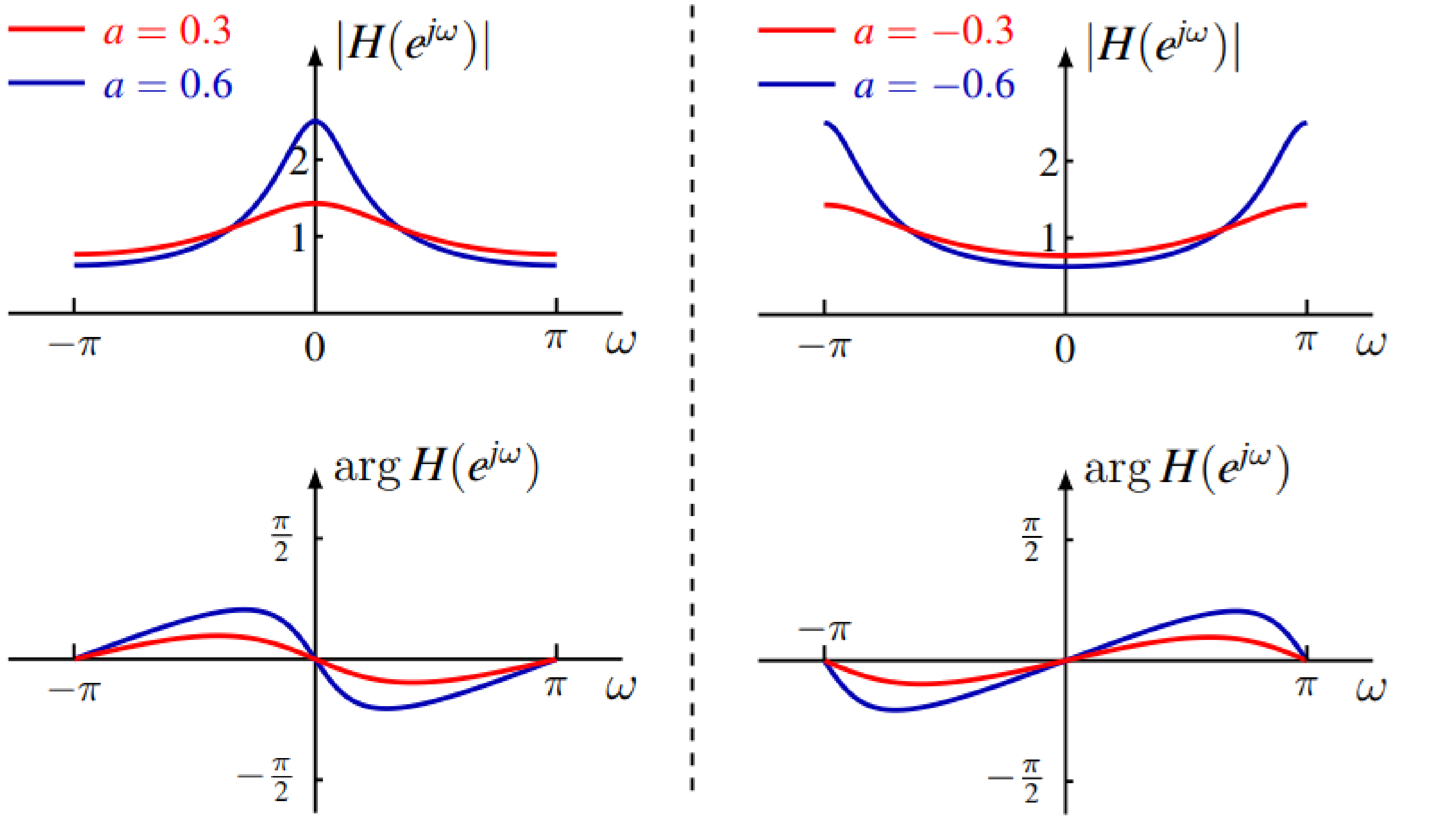
$$\left|H\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 + |a|^2 - 2|a|\cos(\omega - \phi)}}$$

$$\arg H\left(e^{j\omega}\right) = \arctan\frac{-|a|\sin(\omega - \phi)}{1 - |a|\cos(\omega - \phi)}$$

First-order Recursive DT Filters

For $a > 0$, lowpass filter (exponential smoothing)

For $a < 0$, highpass filter



First-order Recursive DT Filters

- **Impulse response** (IIR filter)

$$h[n] = a^n u[n]$$

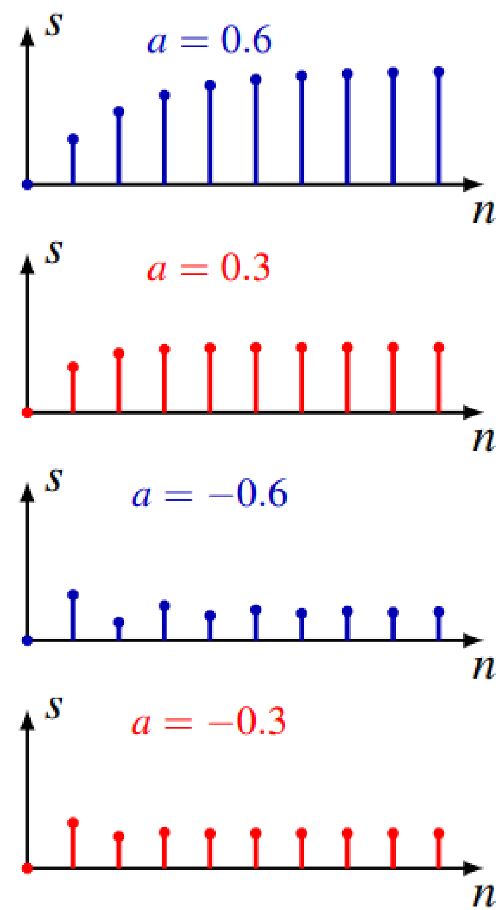
$$H(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

need $|a| < 1$ for convergence

- **Step response**

$$s[n] = h[n] * u[n] = \frac{1 - a^{n+1}}{1 - a} u[n]$$

- **Tradeoff:**
  - larger $|a|$, narrower passband, slower response
  - smaller $|a|$, faster response, broader passband



Moving Average as Lowpass Filter

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- **Impulse response** (FIR filter)

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$

- **Frequency response**

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} e^{-jk\omega} = \frac{e^{j\frac{M_1 - M_2}{2}\omega}}{M_1 + M_2 + 1} \frac{\sin\left(\frac{M_1 + M_2 + 1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

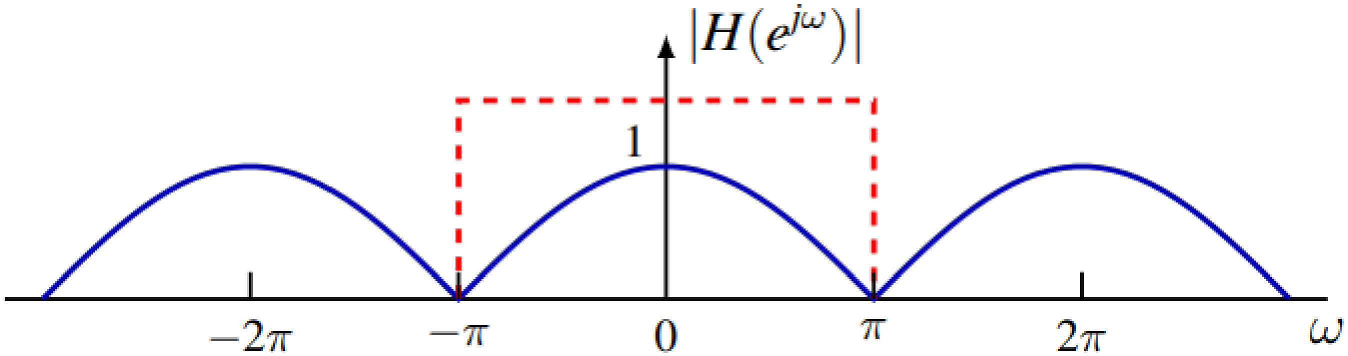
Moving Average as Lowpass Filter

- $M_1 = 0, M_2 = 1$

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

$$H(e^{j\omega}) = e^{-j\omega}\cos\left(\frac{\omega}{2}\right)$$



- Verify $y[n] = x[n]$, if $x[n] = Ke^{j0\cdot n}$ and $y[n] = 0$, if $x[n] = Ke^{j\pi n} = K(-1)^n$

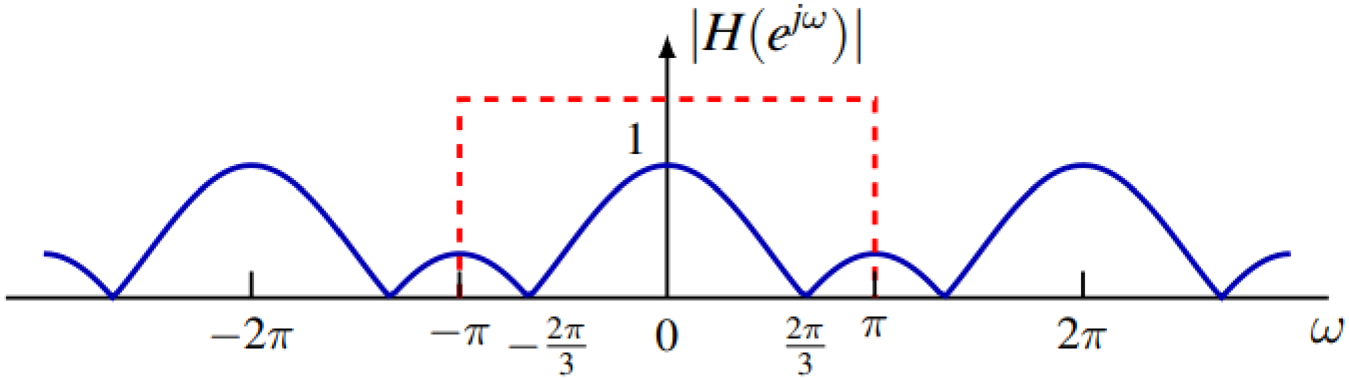
Moving Average as Lowpass Filter

- $M_1 = M_2 = 1$

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

$$h[n] = \frac{1}{3}(\delta[n+1] + \delta[n] + \delta[n-1])$$

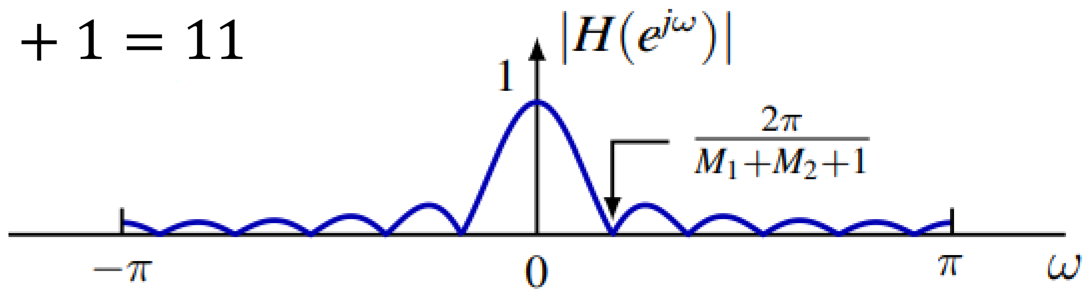
$$H\left(e^{j\omega}\right) = \frac{\sin\left(\frac{3}{2}\omega\right)}{3\sin\left(\frac{\omega}{2}\right)} = \frac{1}{3} + \frac{2}{3}\cos\omega$$



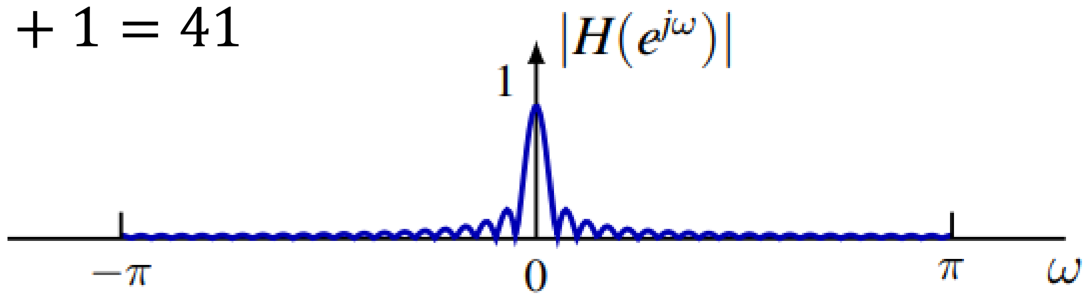
Moving Average as Lowpass Filter

$$H(e^{j\omega}) = \frac{e^{j\frac{M_1-M_2}{2}\omega}}{M_1+M_2+1} \frac{\sin\left(\frac{M_1+M_2+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

- $M_1 + M_2 + 1 = 11$



- $M_1 + M_2 + 1 = 41$



- **Note:** Larger $M_1 + M_2$, narrower passband, smoother output

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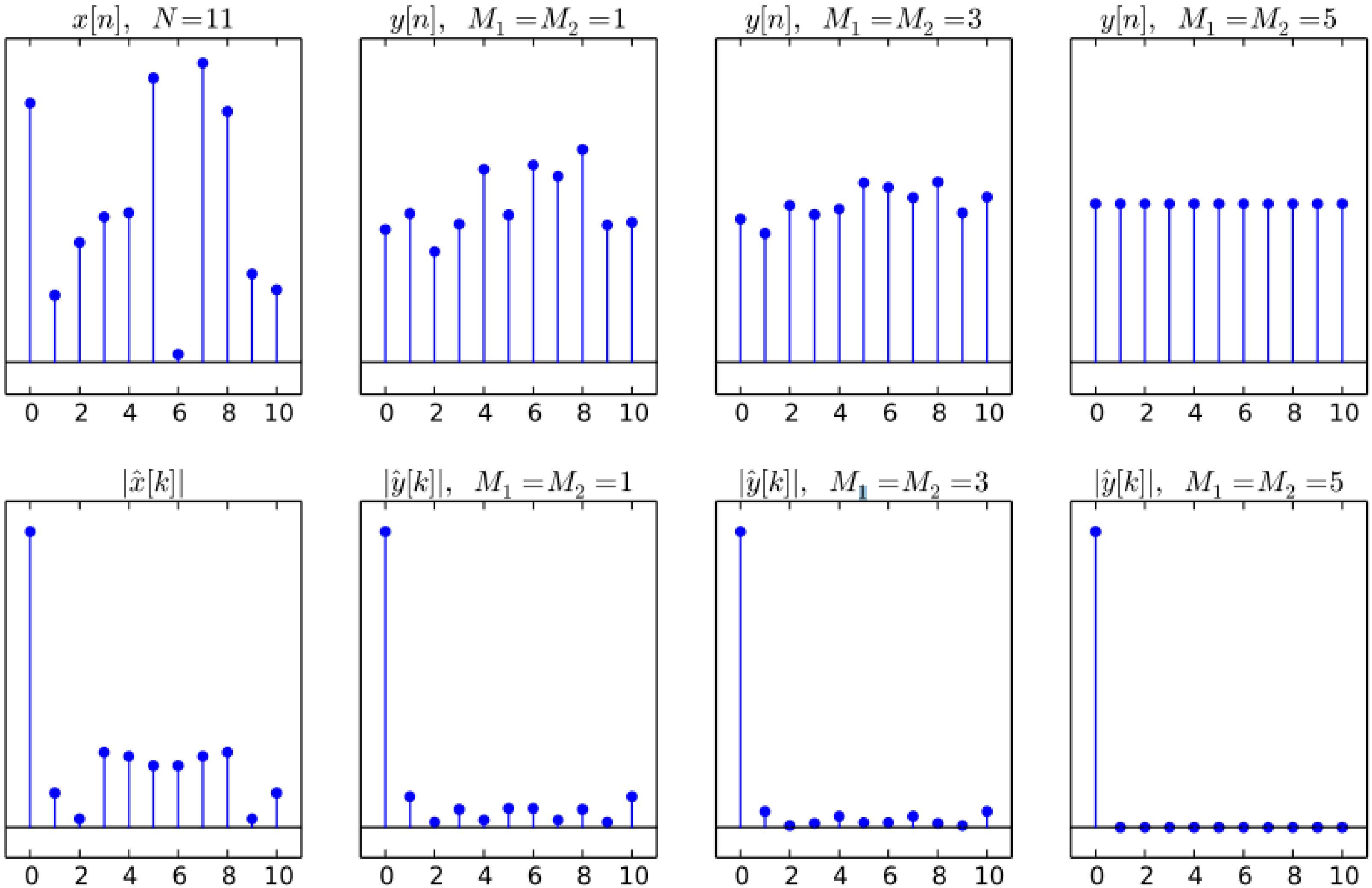
Moving Average as Lowpass Filter



Moving Average as Lowpass Filter

- **noncausal**

$$y_1[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$$

- **causal**

$$y_2[n] = \frac{1}{2M+1}\sum_{k=0}^{2M} x[n-k]$$

Note $y_2[n] = y_1[n-M]$

- **For real-time system**
  - noncausal version not realizable
  - causal version realizable
  - larger M, narrower passband, smoother output, but longer delay, more sluggish response

First Difference as Highpass Filter

- **Scaled first difference**

$$y[n] = \frac{1}{2}(x[n] - x[n-1])$$

- **Impulse response** (FIR filter)

$$h[n] = \frac{1}{2}(\delta[n] - \delta[n-1])$$

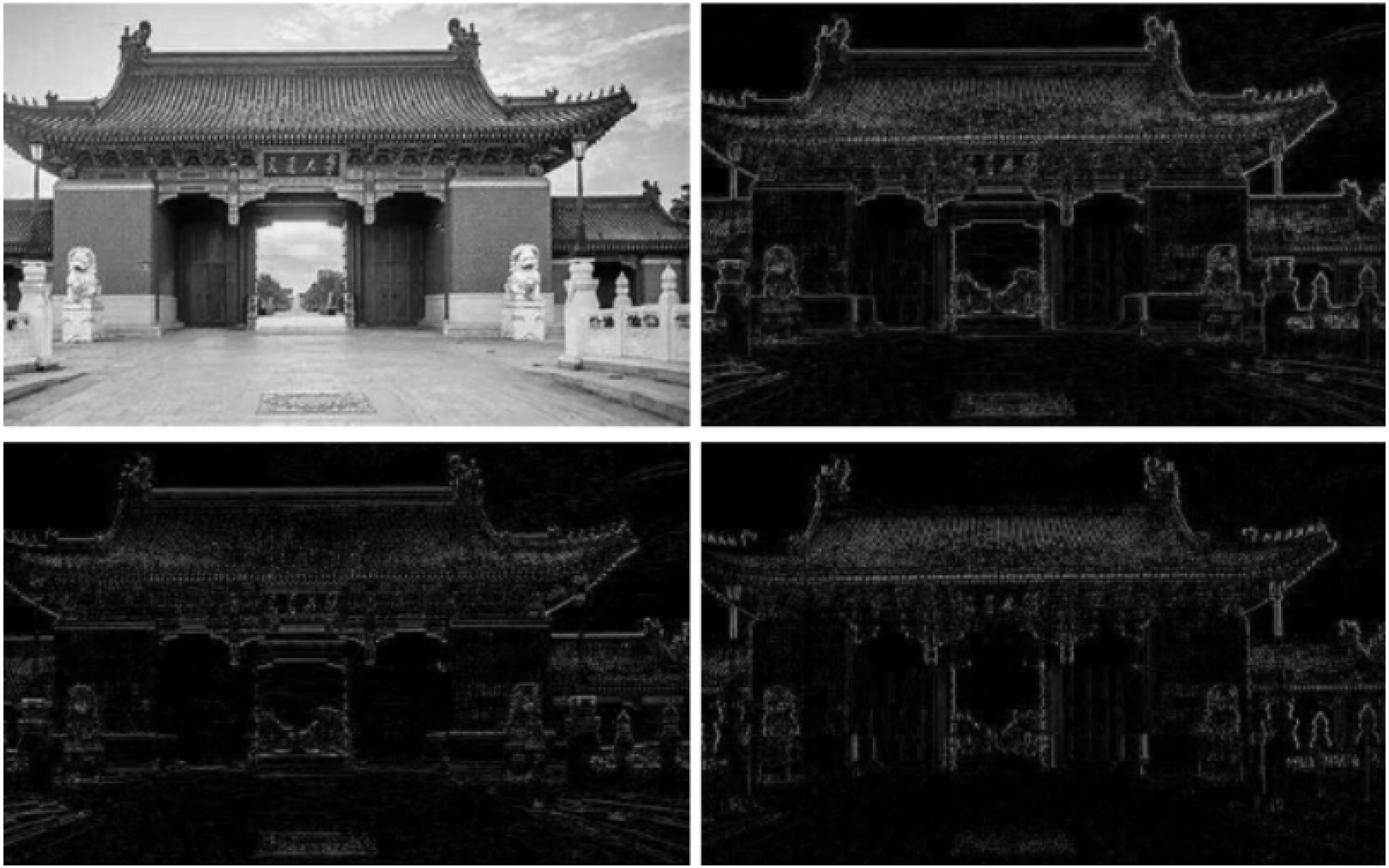
- **Frequency response**

$$H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = je^{-j\omega/2}\sin(\omega/2)$$

$$|H(e^{j\omega})| = |\sin(\omega/2)|$$

- Verify $y[n] = 0$, if $x[n] = Ke^{j0\cdot n}$ and $y[n] = x[n]$, if $x[n] = Ke^{j\pi n} = K(-1)^n$

First Difference for Edge Detection



Q & A

Many Thanks